INSA Centre Val de Loire

ÉCOLE DOCTORALE MIPTIS

Laboratoires LIFO

THÈSE présentée par:

Sajad NAZARI

soutenue le : **16 Décembre 2022**

pour obtenir le grade de : **Docteur de l' INSA Centre Val de Loire**

Discipline/ Spécialité : Informatique

Reasoning with belief functions over Belnap-Dunn logic

THÈSE dirigée par :

M. Benjamin Nguyen	Professeur, INSA Centre Val de Loire (Directeur)
Mme. Sabine Frittella	Maîtresse de Conférences, INSA Centre Val de Loire (Encadrante)

RAPPORTEURS :

Mme. ROUSSET Marie-Christine	Professeure, Université Grenoble-Alpes
M. Pierre SENELLART	Professeur, Ecole Normale Supérieure

JURY :

[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement
[Civilité NOM Prénom]	Titre, établissement

Logo du laboratoire

Logo financeur

Abstract

This work focuses on the representation of uncertain and potentially contradictory information and on how to reason with them. Here we consider crisps and probabilistic information. We build on the work of [KMR21] where the authors define so called non-standard probabilities over Belnap–Dunn logic. Belnap, in *How a computer should think* [Bel19], proposes this propositional logic to model the information a computer might have at its disposal about a statement: the statement might be *true* - that is supported by the evidence -, *false* - that is contradicted by the evidence -, *neither* supported nor contradicted, *both* supported nor contradicted. Building on [KMR21], we propose a modular logical framework to formalize probabilistic reasoning based on inconsistent information [Bíl+20]. Then we generalise their proposal from probabilities to belief functions [Bíl+22]. Belief functions are a generalisation of probabilities that allows to represent incomplete information about a probability distribution. Belief functions are used extensively in order to represent pieces of evidence. Dempster–Shafer theory proposes a rule to aggregate belief functions that is, to obtain a belief function that encodes the information conveyed by many pieces of evidence. Many variations of Dempster–Shafer rule have been proposed, however it remains a benchmark to which these proposals are compared. Therefore, we study the impact of applying Dempster–Shafer rule to belief functions over Belnap–Dunn logic rather than over classical logic. Finally, pursue on that line of research by studying ways to update belief functions over Belnap–Dunn logic [FMN22].

Contents

Introduction

Every day we have to make decisions based on various pieces of information. In fact, to form beliefs about the world, we collect and process data of different origins to provide us with reliable information concerning particular issues. The information at our disposal might be unequivocal (e.g., one sees the rain outside, therefore one knows it is raining), but it might also be incomplete or contradictory. Indeed, we *have no information* whether (to employ the most overused example) the decimal representation of π contains two thousand 9's in a row. On the other hand, we have *both evidence for and against* the efficacy of mirror therapy in phantom pain treatment.

Reasoning with incomplete and inconsistent information. *Incompleteness* of information alone is ever-present when reasoning about data. Applications, such as relational databases, often use many-valued logics to properly account for indefiniteness. Namely, Kleene's three-valued logic [Kle52] became the design choice of SQL and similar systems (the use of Kleene's logic in this context was first proposed by [Cod75], and argued optimal in [CGL18].) In [Bel19], Belnap introduced a four-valued logic with intended database applications (see e.g. [GO15]), which extends Kleene's logic, but also allows to model reasoning with non-trivial *inconsistencies*. Further developed by Dunn [Dun76], Belnap-Dunn four-valued logic BD, also referred to as First Degree Entailment, became a prominent logical framework which encompasses reasoning with both incomplete and inconsistent information. In fact, in the logical context, the logics that can non-trivially reason with contradictory statements are called *paraconsistent* and the ones that allow incomplete information by rejecting the law of excluded middle go under the moniker *paracomplete*. For our purpose, we require a logic that is both paraconsistent and paracomplete. Ideally, this logic should explicitly differentiate between all four types of information an agent can have regarding a statement φ: that φ is only told to be true; that φ is only told to be false; that one was not told whether φ is true or false; and that one was told both that φ is true and that it is false. Originally, in [Bel19], BD was formulated as a four-valued logic with truth table semantics where each value from $\{t, b, n, f\}$ represents the information a computer (a reasoning agent) might have regarding a statement.

- t stands for 'just told True'.
- f stands for 'just told False'.
- b (or 'both') stands for 'told both True and False'.
- n (or 'neither') stands for 'told neither True nor False'.

More precisely, this logic evaluates formulas to Belnap-Dunn square – a lattice built over an extended set of truth values $\{t, f, b, n\}$, where b and n correspond to inconsistent and incomplete information respectively (Figure 1, middle). One of the underlying ideas of this logic is that not only truth, but also amount of information that formulas carry (reflected by the four semantical values) matters. This idea was generalized by introducing the algebraic notion of bilattices by Ginsberg [Gin88] in the context of AI, and studied further in [Riv10; JR12]. Bilattices contain two lattice orders simultaneously: a truth order, and a knowledge (or an information) order. Belnap-Dunn square, the smallest interlaced bilattice, can be seen as the product bilattice of the two-element lattice (Figure 1, left) where the truth-values are pairs of classical values which can be naturally interpreted as representing two independent dimensions of information – positive and negative one[1]. We can understand them as providing positive and negative support for statements independently. It was used to provide the logic with the double-valuation frame semantics by Dunn [Dun76].

[1]This independence assumption has in fact a support in scientific practice – if an experiment confirming a hypothesis fails, does not automatically mean it is rejected.

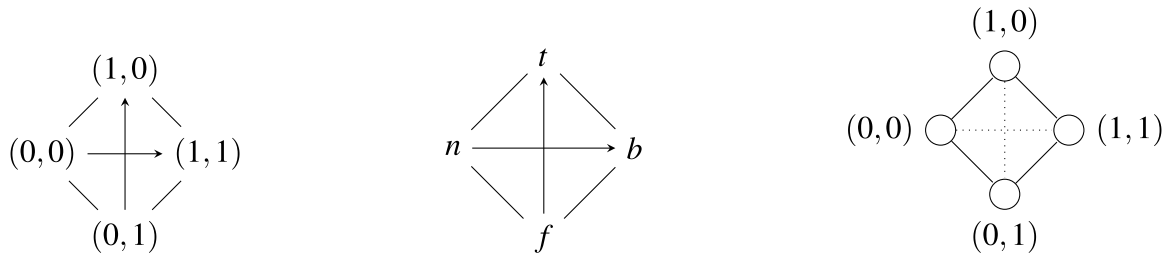


Figure 1: The product bilattice $2 \odot 2$ (left), which is isomorphic to Dunn-Belnap square **4** (middle), and its continuous probabilistic extension (right). Negation flips the values along the horizontal line.

Representation of uncertainty. The information one has is often not only just incomplete or inconsistent, but also bears a degree of uncertainty. This is why one needs a probability theory that accounts for contradictory and incomplete information. In fact, the problem of dealing with inconsistency concerns probabilistic information as well. There are essentially two ways out. One way is to get rid of inconsistencies, the other way is to develop systems with inference rules which can work with inconsistent premises. While on the logic side there are systems providing both kinds of solutions, for example belief revision or paraconsistent logics, the majority of solutions on the probability side go for the first solution – getting rid of inconsistency (cf. the Dempster–Shafer theory of belief functions [Dem68]) – and the attempts of the second kind emerged only relatively recently. Zhou [Zho13] extends the theory of belief functions to the setting of distributive lattices, in particular bilattices and de Morgan lattices, and provides a complete logic to reason about belief functions based on BD. Michael Dunn [Dun10] defines a probabilistic framework over four-valued logic and studies properties of the resulting probabilistic entailment. The idea of an independent account for positive and negative information, underlying the double-valuation semantics of BD, naturally generalizes to non-classical probabilistic extensions of Belnap-Dunn four-valued logic proposed in [KMR21], which we use and extend in this thesis. It generalizes Belnap–Dunn logic in a similar way as classical probability theory generalizes propositional logic, and is referred to as theory of non-standard probabilities. Furthermore, two versions of non-classical probability functions were given a complete axiomatisation. The first is based on independent treatment of positive and negative probabilistic information (the authors call it non-standard probability), the second works with a direct assignment of probabilities to the four values of Belnap–Dunn logic (generalizing an earlier work of M. Dunn on this topic). Both approaches were proven to be equivalent.

Logical formalisms. When it comes to management of uncertainty, probability and other measures of uncertainty can be understood as graded notions, as one tries to quantify the plausibility of unverified events typically over the interval [0,1]. Graded notions are one of the subjects traditionally studied by methods of fuzzy logics. As probability is not truth-functional, it does not admit a straightforward treatment by logical methods. However, one may deal with probability as a modal operator in logical systems (cf. [Ham59]), for example in the systems of modal fuzzy logic [Háj98]. There are two main approaches to probabilistic modalities over classical logic: two-layered and intensional. The two-layered logical formalism introduced in [FHM90; Hal17] separates the non-modal lower language of events from the modal upper language of probabilities. The system divides into three parts: lower level of classical propositional reasoning, reasoning about probabilities consisting of the axioms that characterize probability measures in finite spaces, and the upper level of reasoning about (Boolean combinations of) linear inequalities. Hájek [Háj98] proposed to replace the quantitative reasoning in form of linear inequalities with many-valued reasoning, namely Łukasiewicz logic, in such formalism on the upper level to obtain a fuzzy probability logic for formal reasoning under uncertainty. The graded modality "probably", which can be used to model belief of an agent understood as a kind of subjective probability, is interpreted as a finitely-additive probability on a Boolean algebra of events with values in the real unit interval. Consequently, a class of modal logics for dealing with virtually any uncertainty measure has been covered by the formalism in [CN14].

Structure and contribution of the thesis. My thesis is part of a larger researcher project aiming at extending the framework to encompass reasoning with inconsistent probability information. For this purpose, we first study non-standard probabilities over Belnap-Dunn logic and then we generalise belief functions over Belnap-Dunn logic, and, algebraically, over De Morgan algebras (recall from [Fon97] that Belnap-Dunn logic is the logic of De

Morgan algebras). This part of our work is inspired by [Zho13] which provides treatment of belief functions on distributive lattices. Our goal is to expand that approach to incorporate De Morgan (but still non-classical) negation $\neg$. The detailed structure and contribution of the thesis is as follows:

- **Second Chapter: Preliminaries.** Here we explain the algebraic concepts such as distributive lattices, De Morgan algebras, bilattices, MV-algebras and also Belnap–Dunn logic, its semantics and disjunctive normal forms. Then we explain mass, belief, plausibility functions, Dempster-Shafer combination rule and non-standard probabilities in details and in particular adapt them to De Morgan algebras. In addition we provide some lemmas that we will use later in the thesis.

- **Third Chapter: Reasoning based on inconsistent information.** This chapter which is a reprint of [Bíl+20], is the first steps of the work. We start building the general idea of the thesis and considering the different choices that we can have to attack the problem. We propose an analysis of how belief can be based on information, where the confirmation comes from multiple possibly conflicting sources and is of a probabilistic nature. We use Belnap-Dunn logic and its probabilistic extensions to account for potentially contradictory information on which belief is grounded. We combine it with an extension of Lukasiewicz logic, or a bilattice logic, within a two-layer modal logical framework to account for belief. The two-layer modal logics is proposed for belief of a single agent, belief that is grounded on probabilistic information provided (positive and negative information independently) by multiple sources. The underlying logic of facts or events is chosen to be BD, the upper logic varies between BD and logics derived from Łukasiewicz logic and based on product or bilattice algebras, to systematically account for positive and negative information independently (and thus incompleteness and conflict) on both levels.

- **Forth Chapter: Belief functions over Belnap–Dunn logic.** In this chapter which is a reprint of [Bíl+22], we start working on questions about belief representation and their aggregations on BD logic. First, we generalise belief and plausibility functions over BD logic, and algebraically, over De Morgan algebras. On the other words, we design an expansion of Belnap-Dunn logic with belief and plausibility functions that allows non-trivial reasoning with inconsistent and incomplete probabilistic information. In addition, based on the fact that in the Dempster Shafer combination rule we get rid of contradictory information it is predictable that this rule does give counterintuitive results in presence of contradictory information. As we show this problem can be solved and this rule can be adapted over the frameworks that allows us to have contradictory information.

- **Fifth Chapter: Updating belief functions over Belnap–Dunn logic.** In this chapter which is a reprint of [FMN22], we focus on the natural question which is understanding the behavior of probability, belief and plausibility in the situation where we find a new piece of evidence. In the literature, for example [Hal17], different ways of updating belief and plausibility functions, in the classical framework, is represented. We adapt and interpret those results within the framework of belief and plausibility functions over Belnap-Dunn logic that we have already define in the previous chapters. We present a first approach via a frame semantics of Belnap-Dunn logic. This frame semantics, relying on sets, we can use Bayesian update and Dempster-Shafer combination rule over powerset algebras to define their corresponding updates within the framework of BD logic.

CHAPTER 1 Background and related work

In this chapter, we first recall standard definitions and provide some background information considering lattices, BD logic, Dempster-Shafer theory and non-standard probabilities and fix notational conventions. We also present some lemmas that we will need in the next chapters, and briefly describe some works closest to ours. Note that for a set X, by $P(X)$ we mean powerset of the set X.

1.1 Lattices

Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ and $\mathcal{Q} = \langle Q, \leq_{\mathcal{Q}} \rangle$ be two partially ordered sets (posets). A function $f : \mathcal{P} \to \mathcal{Q}$ is called *monotone* if, for every $x, y \in \mathcal{P}$, $x \leq_{\mathcal{P}} y$ implies that $f(x) \leq_{\mathcal{Q}} f(y)$. It is called *strictly monotone* if $x <_{\mathcal{P}} y$ implies that $f(x) <_{\mathcal{Q}} f(y)$. A structure $\mathcal{L} = \langle L, \vee, \wedge \rangle$, consisting of a set L and two binary, commutative, associative, and idempotent operations, $\vee$ and $\wedge$, on L is called a *lattice* if the operations satisfy the following rules: $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$, for all $x, y \in \mathcal{L}$.

A *bounded lattice* is a tuple $\mathcal{L} = \langle L, \vee, \wedge, \top, \bot \rangle$, such that $\langle L, \vee, \wedge \rangle$ is a lattice and for every element $x \in \mathcal{L} : x \vee \top = \top$ and $x \wedge \bot = \bot$. Obviously, every finite lattice has a least and a greatest element, but we reserve the term 'bounded lattice' for the case when *the lattice signature* contains $\top$ and $\bot$. For bounded lattices, we define $\bigvee \varnothing := \bot$ and $\bigwedge \varnothing := \top$. For finite unbounded lattices $\mathcal{L}$, we define $\bigvee\varnothing := \bigwedge\limits_{l\in\mathcal{L}} l$ and $\bigwedge\varnothing := \bigvee\limits_{l\in\mathcal{L}} l$. A lattice $\mathcal{L}$ is:

- *distributive* if $(x\vee y)\wedge z = (x\wedge z)\vee(y\wedge z)$ holds for all $x,y,z\in\mathcal{L}$;
- *complemented* if $\mathcal{L}$ is bounded and every $x\in\mathcal{L}$ has a *complement*: i.e., for every $x\in\mathcal{L}$, there exists $x'\in\mathcal{L}$, such that $x\wedge x' = \bot$ and $x\vee x' = \top$;
- a *(bounded) De Morgan algebra* if it is (bounded,) distributive, and equipped with an additional unary operation $\neg$ such that $\neg\neg x = x$ and $\neg(x\wedge y) = \neg x\vee\neg y$ for any $x,y\in\mathcal{L}$;
- a *Boolean algebra* if it is a bounded and complemented De Morgan algebra s.t. $\sim$ is its negation and $\sim a$ is the only complement of a.

Convention 1.1.1. *Throughout the manuscript, we denote proper De Morgan negations with $\neg$ and Boolean negations with $\sim$.*

It is well-known that every finite Boolean algebra is isomorphic to the Boolean algebra $\langle P(S),\subseteq\rangle$ for some set S. For a bounded lattice $\langle P(S),\subseteq\rangle$, we have $\vee = \cup$, $\wedge = \cap$, $\top = S$ and $\bot = \varnothing$.

Note that, in bounded De Morgan algebras, the following holds: $\neg(x\vee y) = \neg x\wedge\neg y$, $\neg\top = \bot$ and $\neg\bot = \top$. The law of excluded middle, $\neg x\vee x = \top$, and the principle of explosion, $x\wedge\neg x = \bot$, however, do not always hold.

Definition 1.1.1. *A* logic *is a tuple $\mathsf{L} = \langle\mathscr{L},\vdash\rangle$ with $\mathscr{L}$ being a language over $\{\circ_1,\ldots,\circ_n\}$ and $\vdash\subseteq P(\mathscr{L})\times\mathscr{L}$ where $\vdash$ is reflexive and transitive. A* Lindenbaum algebra of L *(denoted $\mathcal{L}_{\mathsf{L}}$) is a tuple $\langle\mathscr{L}/_{\dashv\vdash},\bullet_1,\ldots,\bullet_n\rangle$ where for each $i\in\{1,\ldots,n\}$ and each $\varphi,\varphi'\in\mathscr{L}$, it holds that $[\varphi\circ_i\varphi'] = [\varphi]\bullet_i[\varphi']$ with $[\varphi]$ being the equivalence class of φ under $\dashv\vdash$.*

Some bilattices and MV algebras For the definition and results about bilattices we refer to [Avr96]. A *bilattice* is an algebra $\mathcal{B} = \langle B,\wedge,\vee,\sqcap,\sqcup,\neg\rangle$ such that the reducts $\langle B,\wedge,\vee\rangle$ and $\langle B,\sqcap,\sqcup\rangle$ are both lattices and the negation $\neg$ is a unary operation satisfying that for every $a,b\in B$,

$$\text{if } a\leq_t b \text{ then } \neg b\leq_t\neg a,\qquad \text{if } a\leq_k b \text{ then } \neg a\leq_k\neg b,\qquad a=\neg\neg a,$$

with $\leq_t$ (resp. $\leq_k$) the order on $\langle B, \wedge, \vee \rangle$ (resp. $\langle B, \sqcap, \sqcup \rangle$) called the truth (resp. knowledge or information) order. So the negation has the interpretation that if a is less true than b then $\neg b$ is less true than $\neg a$ and if your knowledge about a is less than your knowledge about b then still your knowledge about $\neg a$ is less than your knowledge about $\neg b$. A bilattice is *interlaced* if each one of the four operations $\wedge$, $\vee$, $\sqcap$, $\sqcup$ is monotone w.r.t. both orders $\leq_t$ and $\leq_k$ (e.g. if $a \leq_t b$ then $a \wedge c \leq_t b \wedge c$). Bilattices, as well as interlaced bilattices, form a variety. We recall the definition of variety. Let ε be a set of lattice identities (equations), and denote by $Mod\varepsilon$ the class of all lattices that satisfy every identity in ε. A class V of lattices is a lattice variety if $V = Mod\ \varepsilon$ for some set of lattice identities ε. The class of all lattices, is of course a lattice variety since it is equal to $Mod\ \emptyset$. Given an arbitrary lattice $\mathcal{L} = \langle L, \wedge_L, \vee_L \rangle$, we can construct the *product bilattice* $\mathcal{L} \odot \mathcal{L} = \langle L \times L, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ as follows: for all $(a_1, a_2), (b_1, b_2) \in L \times L$,

$$(a_1, a_2) \leq_t (b_1, b_2) \ \text{ iff } a_1 \leq b_1 \text{ and } b_2 \leq a_2$$

$$\neg(a_1, a_2) := (a_2, a_1)$$

$$(a_1, a_2) \wedge (b_1, b_2) := (a_1 \wedge_L b_1, a_2 \vee_L b_2)$$

$$(a_1, a_2) \vee (b_1, b_2) := (a_1 \vee_L b_1, a_2 \wedge_L b_2)$$

$$(a_1, a_2) \sqcap (b_1, b_2) := (a_1 \wedge_L b_1, a_2 \wedge_L b_2)$$

$$(a_1, a_2) \sqcup (b_1, b_2) := (a_1 \vee_L b_1, a_2 \vee_L b_2)$$

$\mathcal{L} \odot \mathcal{L}$ is always an interlaced bilattice, and any interlaced bilattice can be represented as a product billatice: a bilattice $\mathcal{B}$ is interlaced if and only if there is a lattice $\mathcal{L}$ such that $\mathcal{B} \cong \mathcal{L} \odot \mathcal{L}$ [Avr96, Proposition 3.4].

Example 1.1.1. *The smallest interlaced bilattice is the product bilattice of the two-element lattice* $\mathbf{2} \odot \mathbf{2}$ *(Figure 1.1 left). It is isomorphic to Dunn-Belnap square* $\mathbf{4}$ *used as a matrix of truth values for Belnap-Dunn logic (Figure 1.1 middle), with* $\{t, b\}$ *being the designated values.*

Example 1.1.2. *A probabilistic extension of Dunn-Belnap square (Figure 1.1 right) can be seen as based on the product bilattice* $\mathcal{L}_{[0,1]} \odot \mathcal{L}_{[0,1]}$*, where* $\mathcal{L}_{[0,1]} = \langle [0,1], \min, \max \rangle$.

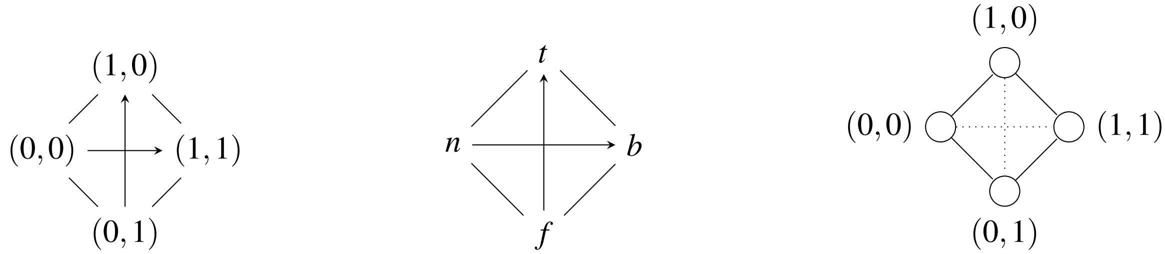


Figure 1.1: The product bilattice $2 \odot 2$ (left), which is isomorphic to Dunn-Belnap square **4** (middle), and its continuous probabilistic extension (right). Negation flips the values along the horizontal line.

Definition 1.1.2. *A* residuated lattice *is an algebra* $\mathcal{L} = \langle L, \wedge_L, \vee_L, \cdot, \backslash, / \rangle$*, where the reduct* $\langle L, \wedge_L, \vee_L \rangle$ *is a lattice,* $\langle L, \cdot \rangle$ *is a semi-group (i.e. the operation* $\cdot$ *is associative) and the residuation properties hold: for all* $a, b, c \in L$*:*

$$a \cdot b \leq c \qquad \textit{iff} \quad b \leq a \backslash c \qquad \textit{iff} \quad a \leq c/b.$$

Example 1.1.3. *1.* $[0,1]_{Ł} = \langle [0,1], \wedge, \vee, \&_{Ł}, \rightarrow_{Ł} \rangle$*, the standard algebra of Łukasiewicz logic, is a residuated lattice, an MV algebra*[1]*, and it generates the variety of MV algebras. (As* $\&_{Ł}$ *is commutative, the two implications coincide.) For all* $a, b \in [0,1]$*, we define a negation* $\sim_{Ł} a := a \rightarrow_{Ł} 0 := 1 - a$*, and the standard operations*

$$a \wedge b := \min\{a, b\}, \qquad\qquad a \&_{Ł} b := \max\{0, a + b - 1\},$$
$$a \vee b := \max\{a, b\}, \qquad\qquad a \rightarrow_{Ł} b := \min\{1, 1 - a + b\}.$$

2. $[0,1]_{Ł}^{op} = ([0,1]^{op}, \vee, \wedge, \oplus_{Ł}, \ominus_{Ł})$ *arises turning the standard algebra upside down, and is isomorphic to the original one. Here, we have* $\sim_{Ł} a := 1 \ominus_{Ł} a$*,*

$$a \oplus_{Ł} b \approx {\sim} a \rightarrow_{Ł} b = \min\{1, a + b\} \qquad a \ominus_{Ł} b \approx {\sim}(a \rightarrow_{Ł} b) = \max\{0, a - b\}.$$

3. Finally, we will consider the product MV algebra $[0,1]_{Ł} \times [0,1]_{Ł}^{op} = ([0,1] \times [0,1]^{op}, \wedge, \vee,$

[1] For more on Łukasiewicz logic and MV algebras (in particular finite standard completeness w.r.t. $[0,1]_{Ł}$) see e.g. [DL11].

$\&, \to \rangle$ *with operations defined pointwise,* $\sim(a_1,a_2) := a \to (0,1) = (\sim a_1, \sim a_2)$*, and:*

$$(a_1,a_2)\&(b_1,b_2) := (a_1 \&_Ł b_1, a_2 \oplus_Ł b_2)$$
$$(a_1,a_2) \to (b_1,b_2) := (a_1 \to_Ł b_1, b_2 \ominus_Ł a_2) = (a_1 \to_Ł b_1, \sim a_2 \&_Ł b_2).$$

As both the projections are surjective homomorphisms of MV algebras, this algebra also generates the variety of MV algebras.

Given a residuated lattice $\mathcal{L} = \langle L, \wedge_L, \vee_L, \cdot, \backslash, / \rangle$ the *product residuated bilattice* [JR12] $\mathcal{L} \odot \mathcal{L} = \langle L \times L, \wedge, \vee, \sqcap, \sqcup, \supset, \subset, \neg \rangle$ is defined as follows: the reduct $\langle L \times L, \wedge, \vee, \sqcap, \sqcup \rangle$ is the product bilattice $\langle L, \wedge_L, \vee_L \rangle \odot \langle L, \wedge_L, \vee_L \rangle$ and, for all $(a_1,a_2),(b_1,b_2) \in L \times L$,

$$(a_1,a_2) \supset (b_1,b_2) := (a_1 \backslash b_1, b_2 \cdot a_1), \qquad (a_1,a_2) \subset (b_1,b_2) := (a_1 / b_1, b_1 \cdot a_2).$$

One can then define the following operations: for all $a,b \in L \times L$,

$$a \to b := (a \supset b) \wedge (\neg a \subset \neg b), \qquad a \leftarrow b := \neg a \to \neg b, \qquad a * b := \neg(b \to \neg a).$$

For any product residuated bilattice, the structure $\langle L \times L, \wedge, \vee, *, \to, \leftarrow, \neg \rangle$ is a residuated bilattice endowed with an involutive negation. If $\cdot$ is commutative (associative), so is $*$.

Example 1.1.4. *The product residuated bilattice arising from the standard MV algebra is the structure* $[0,1]_Ł \odot [0,1]_Ł = \langle [0,1] \times [0,1], \wedge, \vee, \sqcap, \sqcup, \supset, \neg, (0,0) \rangle$ *where:*

$$(a_1,a_2) * (b_1,b_2) := (a_1 \&_Ł b_1, (a_1 \to_Ł b_2) \wedge (b_1 \to_Ł a_2))$$
$$(a_1,a_2) \to (b_1,b_2) := ((a_1 \to_Ł b_1) \wedge (b_2 \to_Ł a_2), a_1 \&_Ł b_2),$$

and $(1,1)$ *acts as the unit of the* $*$*:* $(1,1) * a = a * (1,1) = a$.[2] *We define*

$$\begin{aligned}
\sim a &:= (a \supset (0,0)) \sqcup \neg(\neg a \supset (0,0)) &&= (\sim_Ł a_1, \sim_Ł a_2) \\
a \oplus b &:= (\sim a \supset b) \sqcup \neg(\sim \neg a \supset \neg b) &&= (a_1 \oplus_Ł b_1, a_2 \oplus_Ł b_2) \\
a \ominus b &:= \sim(a \supset b) \sqcap \neg \sim(\neg a \supset \neg b) &&= (a_1 \ominus_Ł b_1, a_2 \ominus_Ł b_2).
\end{aligned}$$

From [JR12], we know that the (isomorphic copies of) product residuated bilattices obtained

[2]Definitions of $\to, *$ match those used in [CDK06] for interval based fuzzy logics, via a transformation given by $(x_1,x_2) \mapsto (x_1, 1-x_2)$ (symmetry across the $(0,0.5)(1,0.5)$ line).

from MV algebras form a variety, and its axiomatization can be obtained by translating the one of MV algebras (in the language of residuated lattices)[3].

1.2 Belnap-Dunn logic

BD logic mentioned in the introduction was introduced by Nuel Belnap in [Bel19]. His main aim was to design a logical system capable of dealing with inconsistent or/and incomplete information. Formulas in Belnap–Dunn logic can be not only True or False, but can also have values Both or Neither. In this section, we are presenting Belnap–Dunn logic (BD) in more details — a propositional logic over $\{\neg, \wedge, \vee\}$ and its conservative extension with constants $\top$ and $\bot$ — BD^*. More formally, we fix a finite (or countable) set $\mathtt{Prop}$ of propositional variables and define complex formulas via the following grammars:

$$\mathscr{L}_{\mathsf{BD}} \ni \varphi \coloneqq p \in \mathtt{Prop} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

and

$$\mathscr{L}^*_{\mathsf{BD}} \ni \varphi \coloneqq p \in \mathtt{Prop} \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

We also define $\mathsf{Lit} = \mathtt{Prop} \cup \{\neg p : p \in \mathtt{Prop}\}$ and denote

$$\mathsf{Var}(\varphi) = \{p \in \mathtt{Prop} : p \text{ occurs in } \varphi\}, \qquad \mathsf{Lit}(\varphi) = \{l \in \mathsf{Lit} : l \text{ occurs in } \varphi\}.$$

BD can be axiomatised using the following axioms from [Pre18]:

$$\begin{array}{cccc} \varphi \wedge \chi \vdash \varphi & \varphi \wedge \chi \vdash \chi & \varphi \vdash \varphi \vee \chi & \chi \vdash \varphi \vee \chi \\ \varphi \vee \psi \vdash \neg\neg\varphi \vee \psi & \varphi, \chi \vdash \varphi \wedge \chi & \neg\neg\varphi \vee \psi \vdash \varphi \vee \psi & \varphi \vee \varphi \vdash \varphi \end{array}$$

$$\varphi \vee (\chi \vee \psi) \vdash (\varphi \vee \chi) \vee \psi \qquad \varphi \wedge (\chi \vee \psi) \vdash (\varphi \wedge \chi) \vee (\varphi \wedge \psi)$$

[3][JR12] hints at the correspondence between subvarieties of residuated lattices and residuated bilattices being categorial. This would mean that the mentioned variety is in fact generated by the product bilattice of the standard MV algebra. One could then use the translation from [JR12] to obtain axiomatics of the logic introduced at the end of Subsection 2.2.1.

$$(\varphi \wedge \chi) \vee (\varphi \wedge \psi) \vdash \varphi \vee (\chi \wedge \psi)$$

$$\neg(\varphi \wedge \chi) \vee \psi \vdash (\neg\varphi \vee \neg\chi) \vee \psi \qquad (\neg\varphi \vee \neg\chi) \vee \psi \vdash \neg(\varphi \wedge \chi) \vee \psi$$

$$\neg(\varphi \vee \chi) \vee \psi \vdash (\neg\varphi \wedge \neg\chi) \vee \psi \qquad (\neg\varphi \wedge \neg\chi) \vee \psi \vdash \neg(\varphi \vee \psi) \vee \psi$$

BD also is completely axiomatized using the following axioms and rules:

$$\varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

$$\varphi \vdash \psi \vee \varphi \qquad \varphi \vdash \varphi \vee \psi$$

$$\varphi \vdash \neg\neg\varphi \qquad \neg\neg\varphi \vdash \varphi$$

$$\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\frac{\varphi \vdash \psi, \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi, \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

$$\frac{\varphi \vdash \chi, \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \qquad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}$$

BD^* can be axiomatised by adding the following axioms:

$$\varnothing \vdash \top \qquad \neg\top \vee \varphi \vdash \varphi \qquad \varnothing \vdash \neg\bot \qquad \bot \vee \varphi \vdash \varphi$$

We will say that φ and ψ are *equivalent*, denoted $\varphi \dashv\vdash_{\mathsf{BD}} \psi$, iff both $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable.

There are several ways to provide semantics for BD (cf., e.g. [OW17]). In addition to the already mentioned truth table semantics, one can treat BD as the logic of De Morgan algebras. Indeed, it is clear from [Fon97, Proposition 2.5] that $\mathcal{L}_{\mathsf{BD}} = \langle \mathscr{L}_{\mathsf{BD}}/_{\dashv\vdash_{\mathsf{BD}}}, \wedge, \vee, \neg \rangle$ is the Lindenbaum algebra of BD and that $\mathcal{L}_{\mathsf{BD}}$ is actually a De Morgan algebra. Hence, its $\neg$-less reduct $\mathcal{L}^{+}_{\mathsf{BD}} = \langle \mathscr{L}_{\mathsf{BD}}/_{\dashv\vdash_{\mathsf{BD}}}, \wedge, \vee \rangle$ is a distributive lattice. The Lindenbaum algebra of

BD^* is the bounded De Morgan algebra $\mathcal{L}_{\mathsf{BD}} = \langle \mathscr{L}_{\mathsf{BD}}/_{\dashv\vdash_{\mathsf{BD}^*}}, \wedge, \vee, \neg \rangle$ and is denoted $\mathcal{L}^*_{\mathsf{BD}}$.

The other way of defining semantic for BD is the frame semantic with two valuations: v^+ and v^- which are intuitively interpreted as support of truth and support of falsity. This approach will allow us to treat BD probabilistically and is also in line with its original motivation (one can think of each state as a source that gives us information).

In this context, $w \vDash^+ \varphi$ can be interpreted as 'source w states that φ is true'. Note that this does not exclude the possibility of w telling that φ is false as well. Neither not stating that φ is false implies that w says that φ is true. One can think of w as being a database that may (or may not) have information about φ. The database may for instance contain both 'Tom's birthday is on February, 29th' and 'Tom's birthday is on March, 1st' and also no information at all whether Tom likes apple pies. If we add constants, $\bot$ represents absurdity or incoherence, a piece of information that the agent rejects without considering it. It is important to note that a contradiction is not absurdity or incoherence: it is perfectly possible that a source provides inconsistent data. Dually, $\top$ is a piece of trivial information: the one that is accepted to be true without questions and does not provide any information. Again, an instance of a classical tautology, say $p \vee \neg p$, is not trivial in this framework for it is possible that a source says nothing on p, nor on its negation. We start by more formal definitions.

Definition 1.2.1 (Belnap–Dunn models)**.** *A Belnap–Dunn model is a tuple $\mathfrak{M} = \langle W, v^+, v^- \rangle$ with $W \neq \varnothing$ and $v^+, v^- : \mathtt{Prop} \to P(W)$.*

Definition 1.2.2 (Frame semantics for BD)**.** *Let $\varphi, \varphi' \in \mathscr{L}_{\mathsf{BD}}$. For a model $\mathfrak{M} = \langle W, v^+, v^- \rangle$, we define notions of $w \vDash^+ \varphi$ and $w \vDash^- \varphi$ for $w \in W$ as follows.*

$$
\begin{array}{cc}
w \vDash^+ p \textit{ iff } w \in v^+(p) & w \vDash^- p \textit{ iff } w \in v^-(p) \\
w \vDash^+ \neg\varphi \textit{ iff } w \vDash^- \varphi & w \vDash^- \neg\varphi \textit{ iff } w \vDash^+ \varphi \\
w \vDash^+ \varphi \wedge \varphi' \textit{ iff } w \vDash^+ \varphi \textit{ and } w \vDash^+ \varphi' & w \vDash^- \varphi \wedge \varphi' \textit{ iff } w \vDash^- \varphi \textit{ or } w \vDash^- \varphi' \\
w \vDash^+ \varphi \vee \varphi' \textit{ iff } w \vDash^+ \varphi \textit{ or } w \vDash^+ \varphi' & w \vDash^- \varphi \vee \varphi' \textit{ iff } w \vDash^- \varphi \textit{ and } w \vDash^- \varphi'
\end{array}
$$

In BD^*, the formulas $\bot$ and $\top$ are interpreted as follows:

$$
w \vDash^+ \top \qquad w \nvDash^- \top \qquad w \nvDash^+ \bot \qquad w \vDash^- \bot
$$

We denote the positive and negative interpretations of a formula as follows:

$$|\varphi|^+ := \{w \in W \mid w \vDash^+ \varphi\} \qquad\qquad |\varphi|^- := \{w \in W \mid w \vDash^- \varphi\}.$$

We will make use of the notions of the *positive extension of a formula* (the set of states supporting it): $|\varphi|^+ = \{w \in W \mid w \vDash^+ \varphi\}$, and analogously the *negative extension of a formula* $|\varphi|^- = \{w \in W \mid w \vDash^- \varphi\}$. Notice that in general a negative extension is not a set theoretical complement of the corresponding positive extension, i.e. $|\varphi|^+ \cup |\varphi|^- \subsetneq S$ (some states might support neither φ nor $\neg\varphi$) and $|\varphi|^+ \cap |\varphi|^- \neq \varnothing$ (states in the intersection support both φ and $\neg\varphi$). Moreover, positive and negative extensions are mutually definable: $|\neg\varphi|^+ = |\varphi|^-$.

Convention 1.2.1. *In the remainder of the text, we will not distinguish between a formula and its equivalence class in the Lindenbaum algebra. Therefore, we will write $|\varphi|^+$ both for the positive interpretation of the formula and for the set of states that satisfy all the formulas in the equivalence class of φ. We will always specify whether φ refers to the formula or to its equivalence class.*

Definition 1.2.3. *We say that a sequent $\varphi \vdash_{\mathsf{BD}} \chi$ is* valid *on $\mathfrak{M} = \langle W, v^+, v^- \rangle$ (denoted, $\mathfrak{M} \vDash [\varphi \vdash_{\mathsf{BD}} \chi]$) iff for any $w \in W$, it holds that:*

- *if $w \vDash^+ \varphi$, then $w \vDash^+ \chi$ as well;*
- *if $w \vDash^- \chi$, then $w \vDash^- \varphi$ as well.*

We drop the subscript when the logic at stake is obvious from the context and we use $\varphi \vdash \chi$ instead of $\varphi \vdash_{\mathsf{BD}} \chi$. A sequent $\varphi \vdash \chi$ is universally valid *iff it is valid on every model. In this case, we will say that φ* entails *χ. For the sake of readability, we avoid subscripts, but we will always specify which logic (*BD *or* BD**) we are considering.*

We also will use another way for presenting semanics for BD logic in the part that we present two-layered logic. In fact, Belnap-Dunn logic evaluates formulas to Belnap-Dunn square – the (de Morgan) lattice **4** built over an extended set of truth values $\{t, f, b, n\}$ (Figure 1.1, middle). The consequence relation of logic BD is given, based on the logical matrix $(\mathbf{4}, F)$

with $F = \{t, b\}$ being the designated values, as

$$\Gamma \vDash_{\mathsf{BD}} \varphi \text{ iff } \forall e \ (e[\Gamma] \subseteq F \rightarrow e(\varphi) \in F).$$

Another frame semantics can also be given for BD. Belnap-Dunn *four-valued model* is a tuple $\langle W, \mathbf{4}, e \rangle$ where W is a set of states and e is a valuation of atomic formulas $e : \mathtt{Prop} \times W \rightarrow \mathbf{4}$. The valuation is extended to formulas of $\mathscr{L}_{\mathsf{BD}}$ using the algebraic operations on $\mathbf{4}$ in the expected way. It is known to be (strongly) complete w.r.t. the algebraic and the double valuation (or 4-valued) frame semantics. BD is also known to be *locally finite*.[4]

In what follows, we present a special version of disjunctive normal forms, X-full DNFs (with X being a set of literals). Intuitively, an X-full DNF of φ lists all possible clauses over X that entail φ in BD. This gives a straightforward connection to frame semantics since each state validates some finite set of literals. Furthermore, X-full DNFs are unique up to permutations of literals and clauses which enables their use as canonical counterparts of a given formula.

Definition 1.2.4 (Clauses and normal forms)**.** *A* conjunctive (resp., disjunctive) clause *is a conjunction (resp., disjunction) of literals and (or) constants. We define the following normal forms of the formulas in languages $\mathscr{L}_{\mathsf{BD}}$ and $\mathscr{L}^{*}_{\mathsf{BD}}$.*

- φ *is in* negation normal form *(NNF) iff it does not contain any of the following subformulas:* $\neg\neg\psi$, $\neg(\psi \vee \psi')$, $\neg(\psi \wedge \psi')$, $\neg\top$, $\neg\bot$;

- φ *is in* disjunctive normal form *iff it is a disjunction of conjunctive clauses;*

- φ *is in* conjunctive normal form *iff it is a conjunction of disjunctive clauses.*

Definition 1.2.5 (X-full disjunctive normal form)**.** *Let $\varphi \in \mathscr{L}_{\mathsf{BD}}$ and let further $X \supseteq \mathsf{Lit}(\varphi)$ be finite and $\neg$-closed, i.e.*

$$\forall p \in \mathsf{Var} : p \in X \Leftrightarrow \neg p \in X$$

[4] It means there are only finitely many (up to inter-derivability) formulas in a fixed finite set of propositional variables. It affects the completeness of the logic in Subsection 2.2.1. More on BD and its properties can be found e.g. in the thesis [Pre18].

A $\bigwedge$-X-clause cl *is a non-empty subset of* X. *An* X-full disjunctive normal form of φ *(*$\mathsf{fDNF}_X(\varphi)$ *or* $\mathsf{fDNF}(\varphi)$ *if there is no risk of confusion) is defined as the disjunction of all* $\bigwedge$-X*-clauses entailing* φ *over* BD*.*

$$\mathsf{fDNF}_X(\varphi) \coloneqq \bigvee_{\mathsf{cl} \vdash_{\mathsf{BD}} \varphi} \mathsf{cl}$$

To define X*-full disjunctive normal form for formulas* $\varphi \in \mathscr{L}^*_{\mathsf{BD}}$*, we need* $\bigwedge$-X-clause *to be a non-empty subset of* X*,* $\bot$*, or* $\top$*. The* X*-full disjunctive normal form of* φ *is denoted* $\mathsf{fDNF}^*_X(\varphi)$ *or* $\mathsf{fDNF}^*(\varphi)$

The next example shows how to transform a formula into its fDNF.

Example 1.2.1. *Let* $X = \{p, \neg p, q, \neg q\}$*. Consider* p *and* $p \wedge q$*. Clearly, both* p *and* $p \wedge q$ *are already in DNF. We now need to add the remaining clauses:*

$$\begin{aligned}
&\mathsf{fDNF}_X(p) = p \vee (p \wedge \neg p) \vee (p \wedge q) \vee (p \wedge \neg q) \vee (p \wedge \neg p \wedge q) \\
&\vee (p \wedge \neg p \wedge \neg q) \vee (p \wedge q \wedge \neg q) \vee (p \wedge \neg p \wedge q \wedge \neg q) \\
&\mathsf{fDNF}_X(p \wedge q) = (p \wedge q) \vee (p \wedge \neg p \wedge q) \vee (p \wedge q \wedge \neg q) \vee (p \wedge \neg p \wedge q \wedge \neg q)
\end{aligned}$$

Observe that $\bigwedge X$ *itself is always present in the* fDNF_X*.*

Definition 1.2.6 (Irredundant disjunctive normal forms: iDNF)**.** *A conjunctive clause is* irredundant *if it contains each literal at most once. A formula* φ *is in* irredundant disjunctive normal *form if it is a disjunction of irredundant conjunctive clauses, and moreover, if* $\varphi = \bigvee_{i \in I} \varphi_i$*, then, for every* $i, j \in I$ *such that* $i \neq j$*,* $\mathsf{Lit}(\varphi_i) \nsubseteq \mathsf{Lit}(\varphi_j)$*.*

Intuitively, no clause of an iDNF implies another one. For example, if $\mathsf{Lit} = \{p, q, \neg p, \neg q\}$ then $(p \wedge q) \vee (p \wedge \neg q)$ is in iDNF but $(p \wedge q) \vee p$ is not.

1.3 Dempster-Shafer Theory

Many generalisations of classical probability theory, such as inner and outer measures [Hal50], belief and plausibility functions [Sha76], upper and lower probabilities [Dem67], have been developed to account for the fact that an agent is not necessarily capable of assigning probabilities to all events. Among the approaches that deal with uncertainty we have chosen Dempster–Shafer theory of belief functions (also called Dempster–Shafer theory of evidence). In this section we explain in details the theory and the concepts used in this theory. This theory first has been introduced by Dempster [Dem67] and Shafer [Sha76] on Boolean algebras and has been generalised to distributive lattices. Since we study this theorem on De Morgan algebras we need to explain the definition and theorems on lattices and note that they all hold for Boolean structures as a particular case. The key concept in Dempster shafer theory is belief function which is inseparable from mass functions. On the other hand, mass functions are indeed special case of Möbius transforms. We first explain Möbius transforms not only for the sake of understanding mass function but also we will need them in some parts of our work. Then after that we provide the interpretation of belief and plausibility functions over De Morgan algebras. We also give some results and lemmas that will be needed later.

Monotone functions on posets and their Möbius transforms It is well-known (cf. [Sta11, Proposition 3.7.1]) that if f is an arbitrary real-valued function on a poset $\mathcal{P} = \langle P, \leq \rangle$, then there exists a unique function g on $\mathcal{P}$, called the *Möbius transform* of f such that:

$$f(x) = \sum_{y \leq x} g(y) \quad \text{iff} \quad g(x) = \sum_{y \leq x} \mu(y,x) \cdot f(y) \tag{1.1}$$

where $\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ is the *Möbius function* defined recursively as follows:

$$\mu(y,x) = \begin{cases} 1 & \text{if } y = x, \\ -\sum\limits_{y \leq t < x} \mu(y,t) & \text{if } y < x, \\ 0 & \text{if } y > x. \end{cases} \tag{1.2}$$

Belief functions, plausibility functions and their associated mass functions We slightly generalize the definitions that were initially proposed in the context of Boolean algebras in order to encompass the case of De Morgan algebras. We need to do this because existing definitions and results consider belief functions on bounded lattices. In our work in different chapters, however, sometimes we study belief functions within the framework of BD which is usually considered without constants $\perp$ and $\top$. Therefore, its associated Lindenbaum algebras are unbounded De Morgan algebras. So we try to distinguish our definitions by the term "general", so we have (general) belief functions and (general) plausibility functions as follows.

Definition 1.3.1 (k-monotonicity)**.** *Let $\mathcal{L}$ be a lattice and $f : \mathcal{L} \to [0,1]$.*

- *f is k-*monotone *if, for every $a_1, \ldots, a_k \in \mathcal{L}$, it holds that*

$$f\left(\bigvee_{1 \leq i \leq k} a_i\right) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot f\left(\bigwedge_{j \in J} a_j\right). \tag{1.3}$$

- *f is* weakly totally monotone*, if f is k-monotone for $k \geq 1$.*

Definition 1.3.2 (k-valuation)**.** *Let $\mathcal{L}$ be a lattice and $f : \mathcal{L} \to [0,1]$.*

- *f is k-*valuation *if, for every $a_1, \ldots, a_k \in \mathcal{L}$, it holds that*

$$f\left(\bigvee_{1 \leq i \leq k} a_i\right) = \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot f\left(\bigwedge_{j \in J} a_j\right). \tag{1.4}$$

- *f is* ∞-valuation*, if f is k-valuation for $k \geq 1$.*

Remark 1.3.1. *The name k-monotone and totally monotone are the ones used in the literature, however notice that k-monotonicity or even totally monotonicity do not imply monotonicity. Therefore, here we call weakly totally monotone the maps that are usually called totally monotone in [Gra09; Zho13]. In fact, these maps were initially introduced by Barthélemy in [Bar00] and referred to as weakly totally monotone. The name k-monotone and totally monotone which come from the literature are counter-intuitive, however notice that k-monotonicity or even totally monotonicity do not imply monotonicity.*

In what follows, we give an example of an ∞-valuation which is not a monotone function. This example can be also considered as a totally monotone function which is not monotone. First we need some lemmas.

Lemma 1.3.1. *Let $n \in \mathbb{N}$. If f is a n-valuation and $f(\perp) = 0$, then it is a k-valuation for every $k \in \mathbb{N}$ such that $1 \leq k \leq n$.*

Proof. We proceed by induction. We know by assumption that f is a n-valuation and we show that for every $k \geq 3$ if f is a k-valuation then it is a $(k-1)$-valuation. Let $(x_1, \ldots, x_{k-1}) \in \mathcal{L}^{k-1}$ and f be a k-valuation. We have

$$\begin{aligned} f(x_1 \vee \ldots \vee x_{k-1}) &= f(x_1 \vee \ldots \vee x_{k-1} \vee \perp) \\ &= \sum_{J \subseteq \{x_1, \ldots x_{k-1}, \perp\}, J \neq \varnothing} (-1)^{|J|+1} f\left(\bigwedge_{y \in J} y\right) \qquad \text{(by induction hypothesis)} \\ &= \sum_{J \subseteq \{x_1, \ldots x_{k-1}\}, J \neq \varnothing} (-1)^{|J|+1} f\left(\bigwedge_{y \in J} y\right). \end{aligned}$$

The last equality holds because one can eliminate the terms containing $\perp$, since $f(\perp) = 0$. Therefore, f is a $(k-1)$-valuation. □

Lemma 1.3.2. *Let $\mathcal{L}$ be a lattice and $|\mathcal{L}| = n \geq 4$, then every $(n-2)$-valuation on $\mathcal{L}$ is an ∞-valuation.*

Proof. By Lemma 1.3.1 we just need to prove that it is a k-valuation for every $k \geq n-2$. We proceed by induction. By assumption f is a $(n-2)$-valuation. Now we show that for every $k \geq n-2$ if f is a k-valuation on $\mathcal{L}$ it is a $(k+1)$-valuation. Let $(a_1, \ldots, a_{k+1}) \in \mathcal{L}^{k+1}$, then we have the following cases:

(a) there exist $i \neq j$ such that $a_i = a_j$. W.l.g. we can assume that $a_k = a_{k+1}$:

$$\begin{aligned} f(a_1 \vee \ldots \vee a_{k+1}) &= f(a_1 \vee \ldots \vee a_k) \\ &= \sum_{J \subseteq \{1, \ldots, k\}, J \neq \varnothing} (-1)^{|J|+1} f\left(\bigwedge_{j \in J} a_j\right) \qquad (f \text{ is a } k\text{-valuation}) \\ &= \sum_{J \subseteq \{1, \ldots k+1\}, J \neq \varnothing} (-1)^{|J|+1} f\left(\bigwedge_{j \in J} a_j\right) \end{aligned}$$

the last equality holds since in the last sum, we have some new summands which are can be divided in two: those who contains $k+1$ and k we call them $\mathscr{A}$ and those containing

$k+1$ and not k which we denote by $\mathscr{B}$. The summand $(-1)^{|J|+1} f\left(\bigwedge_{j\in J} a_j\right)$'s, where $J \in \mathscr{B}$, will be eliminated by the summand $(-1)^{|J'|+1} f\left(\bigwedge_{j\in J'} a_j\right)$'s where $J' = J \cup \{k\} \in \mathscr{A}$. So all the new elements will be eliminated pairwise.

(b) a_i's are pairwise distinct and w.l.g we have two cases:

(1) $a_1 = \perp$ then

$$
\begin{aligned}
f(a_1 \vee \ldots \vee a_{k+1}) &= f(a_2 \vee \ldots \vee a_{k+1}) \\
&= \sum_{J\subseteq\{2,\ldots,k+1\}, J\neq\varnothing} (-1)^{|J|+1} f\left(\bigwedge_{j\in J} a_j\right) && (f \text{ being } k\text{-valuation}) \\
&= \sum_{J\subseteq\{1,\ldots,k+1\}, J\neq\varnothing} (-1)^{|J|+1} f\left(\bigwedge_{j\in J} a_j\right) && \left(f\left(\bigwedge_{j\in J} a_j\right) = f(\perp) = 0 \text{ if } 1 \in J\right)
\end{aligned}
$$

(2) $a_1 = \top$ then the non-empty elements of $P(J)$ where $J = \{1, \ldots, k+1\}$ consist of two collections: those not containing 1, say $\mathscr{A}$, and those containing 1, say $\mathscr{B}$. And $\theta : \mathscr{A} \to \mathscr{B} \setminus \{\{1\}\}$ such that $\theta(T) = T \cup \{1\}$ is a bijection. Notice that

$$(-1)^{|T|+1} f(\bigwedge_{j\in T} a_j) + (-1)^{|\theta(T)|+1} f(\bigwedge_{j\in\theta(T)} a_j) = 0,$$

so in the sum $\sum_{J\subseteq\{1,\ldots,k+1\}, J\neq\varnothing} (-1)^{|J|+1} f\left(\bigwedge_{j\in J} a_j\right)$ the summands based on $T \subseteq J$, where $T \in \mathscr{A}$, eliminate the summands based on $T \cup \{1\} \subseteq J$ so the only remaining term will be $f(a_1)$ so:

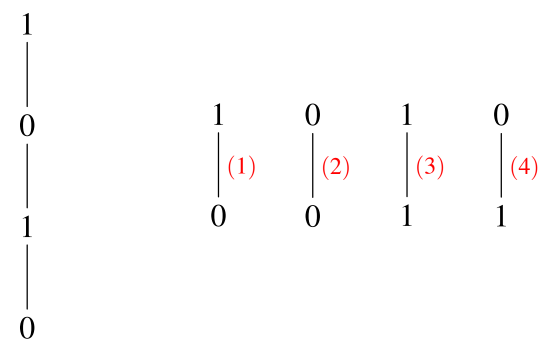
$$
\begin{aligned}
\sum_{J\subseteq\{1,\ldots,k+1\}, J\neq\varnothing} (-1)^{|J|+1} f\left(\bigwedge_{j\in J} a_j\right) &= f(a_1) = f(\top) = 1 \\
&= f(a_1 \vee \ldots \vee a_{k+1})
\end{aligned}
$$

So by Lemma 1.3.1 f is an ∞-valuation.

$\square$

Example 1.3.1. *Consider the following 4-element chain (say $\perp \leq a \leq b \leq \top$) and the function*

f defined as $f(\bot) = 0, f(a) = 1, f(b) = 0$ and $f(\top) = 1$ which we show on the figure below on the left.



This lattice is a distributive lattice and for showing that the function f is a ∞-valuation we just need to show that it is a 2-valuation (based on or Lemma 1.3.2). For checking whether this function is a 2 valuation we have 6 cases (picking 2 distinct elements among 4) which will boil down to the four cases in the left part of the above figure. In all this cases we can see that $f(a \vee b) = f(a) + f(b) - f(a \wedge b)$, in fact we have the following computations:

Case 1: $1 = 0 + 1 - 0$

Case 2: $0 = 0 + 0 - 0$

Case 3: $1 = 1 + 1 - 1$

Case 4: $0 = 0 + 1 - 1$

So this function is a 2-valuation and so by Lemma 1.3.2 is a ∞-valuation (therefore totally monotone), but not a monotone function.

In what follows, we introduce definitions and properties about belief and plausibility functions.

Definition 1.3.3 ((General) belief functions)**.** *Let $\mathcal{L}$ be a lattice. A function* $\texttt{bel} : \mathcal{L} \to [0,1]$ *is called a* general belief function *if the following conditions hold:*

- $\texttt{bel}$ *is monotone with respect to $\mathcal{L}$, that is, for every $x, y \in \mathcal{L}$, if $x \leq_{\mathcal{L}} y$, then* $\texttt{bel}(x) \leq \texttt{bel}(y)$*,*

- $\texttt{bel}$ *is weakly totally monotone, that is, for every* $k \geq 1$ *and every* $a_1, \ldots, a_k \in \mathcal{L}$*, it holds that*

$$\texttt{bel}\left(\bigvee_{1\leq i\leq k} a_i\right) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \texttt{bel}\left(\bigwedge_{j\in J} a_j\right). \tag{1.5}$$

A general belief function $\texttt{bel}$ *on a bounded lattice* $\mathcal{L}$ *is called* belief function *if* $\texttt{bel}(\perp) = 0$ *and* $\texttt{bel}(\top) = 1$.

Definition 1.3.4 ((General) mass function)**.** *Let* $S \neq \varnothing$*. A* general mass function *on* S *is a function* $\mathtt{m} : S \to [0,1]$ *such that* $0 \leq \sum_{x\in S} \mathtt{m}(x) \leq 1$*. A* mass function *on* S *is a function* $\mathtt{m} : S \to [0,1]$ *such that* $\sum_{x\in S} \mathtt{m}(x) = 1$.

The following theorem, [Zho13, Theorem 2.8], presents another characterisation of belief functions which is based on their mass assignment.

Theorem 1.3.1. [Zho13, Theorem 2.8] *Let* $\mathcal{L}$ *be a finite bounded lattice and* $f : \mathcal{L} \to [0,1]$ *be a monotone function such that* $f(\top) = 1$ *and* $f(\perp) = 0$*. Let further,* g *be the Möbius transform of* f*. Then, the following statements are equivalent:*

1. f *is weakly totally monotone (that is,* f *is a belief function);*

2. g *is a mass function.*

Notice that [Zho13, Theorem 2.8] states that f is weakly totally monotone iff g is non-negative, but it is immediate to prove that g is indeed a mass function.

In many parts of our work, we will work with the generalized notions, general belief function and general mass function. Therefore, we will need the following theorem. which is used to prove Theorem 1.3.2. We can look at it as a characterisation of general belief function based on their Möbius transform. But before that we need the following lemma.

Lemma 1.3.3. [Zho13, Lemma 2.6] *Let* $\mu : P(X) \to \mathbb{R}$ *be a function such that:* $\mu(\varnothing) = 0$ *and* $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ *for every* $A, B \in P(X)$*. Then, we have:*

$$\mu\left(\bigcup_{i=1}^{n} R_i\right) = \sum_{J\subseteq\{1,\ldots,n\}, J\neq\varnothing} (-1)^{|J|+1} \mu\left(\bigcap_{j\in J} R_i\right). \tag{1.6}$$

Theorem 1.3.2. *Let $\mathcal{L}$ be a finite lattice, $f : \mathcal{L} \to [0,1]$, and g, the Möbius transform of f. If f is both monotone and weakly totally monotone, then g is a general mass function.*

Proof. The proof is similar to the proof of Theorem 1.3.1. The differences are the following: $\bot$ and $\top$ are not in the signature of the lattice $\mathcal{L}$, we do not require the lattice to be distributive, and f is not required to be a belief function, therefore g is not necessarily a mass function.

Let $a \in \mathcal{L}$. First, we show that $g(a) \geq 0$. Notice that, since $f(a) = \sum_{b \leq a} g(b)$, we have $g(a) = f(a) - \sum_{b<a} g(b)$. Let $A = \{x \in \mathcal{L} \mid x < a\}$. Recall that $A = \bigcup_{b<a} \downarrow b$, where $\downarrow b = \{x \in \mathcal{L} \mid x \leq b\}$. Let $\mu : P(\mathcal{L}) \to [0,1]$ be such that $\mu(A) := \sum_{x \in A} g(x)$. Notice that μ is additive. The following chain of equalities holds:

$$
\begin{aligned}
\sum_{b<a} g(b) = \sum_{x \in A} g(x) & \\
&= \mu(A) && \text{(definition of } \mu\text{)} \\
&= \sum_{J \subseteq A, J \neq \varnothing} (-1)^{|J|+1} \mu\left(\bigcap_{b \in J} \downarrow b\right) && (A = \textstyle\bigcup_{b<a} \downarrow b \text{ and } \mu \text{ satisfies (1.6)}) \\
&= \sum_{J \subseteq A, J \neq \varnothing} (-1)^{|J|+1} \left(\sum_{x \in \bigcap_{b \in J} \downarrow b} g(x)\right) && \text{(definition of } \mu\text{)} \\
&= \sum_{J \subseteq A, J \neq \varnothing} (-1)^{|J|+1} \left(\sum_{x \in \downarrow \bigwedge \{x \mid x \in J\}} g(x)\right) && (\downarrow a \cap \downarrow b = \downarrow (a \wedge b)) \\
&= \sum_{J \subseteq A, J \neq \varnothing} (-1)^{|J|+1} f\left(\bigwedge_{x \in J} x\right) && \text{(definition of } f \text{ and } \downarrow\text{)}
\end{aligned}
$$

Notice that, if there is no element smaller than a, we have $f(a) = g(a)$ and $g(a) \geq 0$ (because f is positive). Otherwise, the following chain of inequalities follows from the equality above and because f is monotone and weakly totally monotone.

$$
f(a) \geq f(\bigvee_{b<a} b) \geq \sum_{J \subseteq A, J \neq \varnothing} (-1)^{|J|+1} f\left(\bigwedge_{x \in J} x\right) = \sum_{b<a} m(b).
$$

Therefore, $g(a) = f(a) - \sum_{b<a} g(b) \geq 0$ as required.

Since $\mathcal{L}$ is a finite lattice, it has a unique maximal element t. Since g is the Möbius transform of f, we have $\sum_{x \in \mathcal{L}} g(x) = \sum_{x \leq t} g(x) = f(t)$. Therefore, since g is non-negative, $0 \leq \sum_{x \in \mathcal{L}} g(x) \leq 1$ and $g : \mathcal{L} \to [0,1]$, that is, g is a general mass function as required. $\square$

The next lemma follows from Theorems 1.3.1 and 1.3.2.

Lemma 1.3.4 (Mass function associated to a (general) belief function)**.** *Let $\mathcal{L}$ be a finite lattice and* $\mathtt{bel} : \mathcal{L} \to [0,1]$ *a general belief function. Then, there is a general mass function* $\mathtt{m}_{\mathtt{bel}} : \mathcal{L} \to [0,1]$*, called the* mass function associated to $\mathtt{bel}$*, such that, for every $x \in \mathcal{L}$,*

$$\mathtt{bel}(x) = \sum_{y \leq x} \mathtt{m}_{\mathtt{bel}}(y). \tag{1.7}$$

If $\mathtt{bel}$ *is a belief function, then* $\sum_{y \in \mathcal{L}} \mathtt{m}_{\mathtt{bel}}(y) = 1$.

In the literature on belief functions, the Möbius transform of a belief function $\mathtt{bel}$ (i.e., g from (1.1)) is denoted by $m_{\mathtt{bel}}$ and called *the mass function associated with* $\mathtt{bel}$. In our work, we focus on (general) belief functions, therefore we always refer to the Möbius transforms of a (general) belief functions as its (general) mass function.

Definition 1.3.5 ((General) plausibility functions)**.** *Let $\mathcal{L}$ be a lattice.* $\mathtt{pl} : \mathcal{L} \to [0,1]$ *is called a* general plausibility function *if the following conditions hold:*

- $\mathtt{pl}$ *is monotone with respect to $\mathcal{L}$,*
- *for every $k \geq 1$ and every $a_1, \ldots, a_k \in \mathcal{L}$, it holds that*

$$\mathtt{pl}\left(\bigwedge_{1 \leq i \leq k} a_i\right) \leq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{pl}\left(\bigvee_{j \in J} a_j\right). \tag{1.8}$$

Let $\mathcal{L}$ be a bounded lattice and $\mathtt{pl}$ *a general plausibility function on $\mathcal{L}$.* $\mathtt{pl}$ *is called* plausibility function *if* $\mathtt{pl}(\bot) = 0$ *and* $\mathtt{pl}(\top) = 1$.

Lemma 1.3.5 (Mass function associated to a (general) plausibility function)**.** *Let $\mathcal{L}$ be a De Morgan algebra, and* $\mathtt{pl} : \mathcal{L} \to [0,1]$ *a general plausibility function. Then, the function* $\mathtt{bel}_{\mathtt{pl}} : \mathcal{L} \to [0,1]$ *such that* $\mathtt{bel}_{\mathtt{pl}}(x) = 1 - \mathtt{pl}(\neg x)$ *is a general belief function. We denote* $\mathtt{m}_{\mathtt{pl}}$ *the mass function associated to* $\mathtt{bel}_{\mathtt{pl}}$ *and we call* $\mathtt{m}_{\mathtt{pl}}$ *the* mass function associated to $\mathtt{pl}$. *Then*

$$\mathtt{pl}(x) = 1 - \sum_{y \leq \neg x} \mathtt{m}_{\mathtt{pl}}(y). \tag{1.9}$$

Moreover, if $\mathtt{pl}$ *is a plausibility function, then* $\mathtt{bel}_{\mathtt{pl}}$ *is a belief function.*

Proof. Consider the function $\mathtt{bel}_{\mathtt{pl}}(x) = 1 - \mathtt{pl}(\neg x)$. Since $0 \leq \mathtt{pl}(\neg x) \leq 1$, then $0 \leq \mathtt{bel}_{\mathtt{pl}}(x) \leq 1$. Therefore, $\mathtt{bel}_{\mathtt{pl}}$ is well-defined. Notice that, since $\neg$ is order-reversing and $\mathtt{pl}$ is order-preserving, $\mathtt{bel}_{\mathtt{pl}}$ is order-preserving. For every $a_1, \ldots, a_k \in \mathcal{L}$, we have

$$
\begin{aligned}
\mathtt{pl}\left(\neg \bigvee_{1\leq i\leq k} a_i\right) &= \mathtt{pl}\left(\bigwedge_{1\leq i\leq k} \neg a_i\right) && (\neg \text{ is a De Morgan negation})\\
&\leq \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{pl}\left(\bigvee_{j\in J} \neg a_j\right) && (\mathtt{pl} \text{ is a general plausibility function})\\
&= \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{pl}\left(\neg \bigwedge_{j\in J} a_j\right) && (\neg \text{ is a De Morgan negation})
\end{aligned}
$$

In addition, we have[5]

$$
\begin{aligned}
\sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1} &= (-1)\cdot \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}} (-1)^{|J|} = (-1)\cdot \sum_{1\leq k\leq n} (-1)^k \binom{n}{k}\\
&= (-1)\cdot\left(\sum_{0\leq k\leq n} (-1)^k \binom{n}{k} - 1\right) \qquad \left(\text{since } \textstyle\sum_{0\leq k\leq n}(-1)^k \binom{n}{k} = 0\right)\\
&= 1
\end{aligned}
$$

[5]Recall that $\binom{n}{k}$ denotes the binomial coefficient, that is, the number of ways to choose an (unordered) subset of k elements from a fixed set of n elements.

Therefore,

$$\begin{aligned}
\mathtt{bel}_{\mathtt{pl}}\left(\bigvee_{1\leq i\leq k} a_i\right) &= 1-\mathtt{pl}\left(\neg\bigvee_{1\leq i\leq k} a_i\right)\\
&\geq \left(\sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}}(-1)^{|J|+1}\right)-\left(\sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}}(-1)^{|J|+1}\cdot\mathtt{pl}\left(\neg\bigwedge_{j\in J}a_j\right)\right)\\
&= \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}}\left((-1)^{|J|+1}-(-1)^{|J|+1}\cdot\mathtt{pl}\left(\neg\bigwedge_{j\in J}a_j\right)\right)\\
&= \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}}(-1)^{|J|+1}\cdot\left(1-\mathtt{pl}\left(\neg\bigwedge_{j\in J}a_j\right)\right)\\
&= \sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}}(-1)^{|J|+1}\cdot\mathtt{bel}_{\mathtt{pl}}\left(\bigwedge_{j\in J}a_j\right).
\end{aligned}$$

Therefore $\mathtt{bel}_{\mathtt{pl}}$ is order-preserving and k-monotone for every $k\geq 1$. Hence, $\mathtt{bel}_{\mathtt{pl}}$ is a general belief function. In addition, if $\mathtt{pl}$ is a plausibility function, then $\mathtt{bel}(\bot)=1-\mathtt{pl}(\top)=0$ and $\mathtt{bel}(\top)=1-\mathtt{pl}(\bot)=1$. Therefore, $\mathtt{bel}$ is a belief function. Let $\mathtt{m}_{\mathtt{pl}}$ be the mass function associated to $\mathtt{bel}_{\mathtt{pl}}$, then we have $\mathtt{pl}(x)=1-\mathtt{bel}(\neg x)=1-\sum_{y\leq\neg x}\mathtt{m}_{\mathtt{pl}}(y)$. □

Lemma 1.3.6. *Let $\mathcal{L}$ be a De Morgan algebra and $\mathtt{bel}:\mathcal{L}\to[0,1]$ a general belief function. Then, the function $\mathtt{pl}:\mathcal{L}\to[0,1]$ such that $\mathtt{pl}(x)=1-\mathtt{bel}(\neg x)$ is a general plausibility function. If $\mathtt{bel}$ is a belief function, then $\mathtt{pl}$ is a plausibility function.*

Proof. The proof is similar to the proof of lemma 1.3.5. We only detail the proof that $\mathtt{pl}$ satisfies equation (1.8) for every $k\geq 1$. Let $a_1,\ldots,a_k\in\mathscr{L}$. Recall that

$$\mathtt{bel}\left(\bigvee_{1\leq i\leq n} a_i\right)\geq\sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}}(-1)^{|J|+1}\cdot\mathtt{bel}\left(\bigwedge_{j\in J}a_j\right).$$

Therefore, we have the following chain of inequalities.

$$\mathtt{bel}\left(\bigvee_{1\le i\le n}\neg a_i\right) \geq \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{bel}\left(\bigwedge_{j\in J}\neg a_j\right)$$

$$-\mathtt{bel}\left(\neg\bigwedge_{1\le i\le n} a_i\right) \leq -\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{bel}\left(\neg\bigvee_{j\in J} a_j\right).$$

Since, $\neg$ is a De Morgan negation and multiplication by (-1) reverses the inequality, we have

$$-\mathtt{bel}\left(\neg\bigwedge_{1\le i\le n} a_i\right) \leq \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \left(-\mathtt{bel}\left(\neg\bigvee_{j\in J} a_j\right)\right)$$

$$\mathtt{pl}\left(\bigwedge_{1\le i\le n} a_i\right)-1 \leq \sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\left(\mathtt{pl}\left(\bigvee_{j\in J} a_j\right)-1\right)$$

(because $-\mathtt{bel}(\neg x) = \mathtt{pl}(x) - 1$)

$$\mathtt{pl}\left(\bigwedge_{1\le i\le n} a_i\right)-1 \leq \left(\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{pl}\left(\bigvee_{j\in J} a_j\right)\right) - \left(\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\right)$$

$$\mathtt{pl}\left(\bigwedge_{1\le i\le n} a_i\right)-1 \leq \left(\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{pl}\left(\bigvee_{j\in J} a_j\right)\right) + \left(\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|}\right)$$

$$\mathtt{pl}\left(\bigwedge_{1\le i\le n} a_i\right)-1 \leq \left(\sum_{\substack{J\subseteq\{1,\dots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{pl}\left(\bigvee_{j\in J} a_j\right)\right) + \left(\sum_{1\le k\le n} (-1)^{k}\binom{n}{k}\right).$$

Since, because $\sum_{0\leq k\leq n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} = 0 = 1 + \sum_{1\leq k\leq n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}$, we get

$$\mathtt{pl}\left(\bigwedge_{1\leq i\leq n} a_i\right) - 1 \leq \left(\sum_{\substack{J\subseteq\{1,\ldots,n\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{pl}\left(\bigvee_{j\in J} a_j\right)\right) - 1$$

$$\mathtt{pl}\left(\bigwedge_{1\leq i\leq k} a_i\right) \leq \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{pl}\left(\bigvee_{j\in J} a_j\right),$$

as required. □

The following lemma will be useful for the proof of theorem 3.4.1. Here $\mathcal{B}_{\mathcal{L}}$ is the intersection of all Boolean algebras containing $\mathcal{L}$:

Lemma 1.3.7. *[Zho13, Lemma 3.7] Let $\mathcal{L}$ be a finite distributive lattice, and $\mathcal{B}_{\mathcal{L}}$ the Boolean algebra generated by $\mathcal{L}$. Any (general) belief function* $\mathtt{bel}$ *on $\mathcal{L}$ can be extended to a belief function* $\mathtt{bel}'$ *on $\mathcal{B}_{\mathcal{L}}$ in the sense that, for any $x\in\mathcal{L}$, $\mathtt{bel}'(x) = \mathtt{bel}(x)$.*

Proof. If $\mathtt{bel}$ is a belief function, then we use [Zho13, Lemma 3.7]. Assume that $\mathtt{bel}$ is a general belief function on a finite distributive lattice $\mathcal{L} = \langle L, \vee, \wedge\rangle$. We consider $\mathtt{bel}^*$ the extension of $\mathtt{bel}$ to the distributive lattice $\mathcal{L}^*$ obtained by adding a top and a bottom element to $\mathcal{L}$. We define $\mathtt{bel}^*(\bot) = 0$ and $\mathtt{bel}^*(\top) = 1$. This new lattice is again a finite distributive lattice. By applying [Zho13, Lemma 3.7] to $\mathtt{bel}^*$, we obtain a belief function $\mathtt{bel}'$ on $\mathcal{B}_{\mathcal{L}^*}$ such that $\mathtt{bel}'(x) = \mathtt{bel}(x)$, for every $x\in\mathcal{L}$. □

1.4 Non-standard probabilities

Probability is the most traditional measure of uncertainty. It is usually introduced as a measure on a Boolean algebra, but it can also be defined as a function on formulas of classical logic satisfying the following axioms:

- $\mathtt{p}(\top) = 1$ (normalisation);

- if $\varphi \vdash_{CL} \psi$ then $\mathtt{p}(\varphi) \leq \mathtt{p}(\psi)$ (monotonicity);

- $\mathtt{p}(\varphi \vee \psi) = \mathtt{p}(\varphi) + \mathtt{p}(\psi)$ for $\varphi \wedge \psi = \perp$ (additivity).

This definition is equivalent to introducing probability on the Lindenbaum algebra of the classical propositional logic using Kolmogorov's axioms. There are various attempts in the literature to define probabilities on structures more general than Boolean algebras. The main purpose of introducing probability measure over BD in [KMR21] was to enrich the framework of Belnap–Dunn logic designed to be able to capture incomplete and/or inconsistent information with an uncertainty measure. The framework is based on the notion of a probabilistic BD model, which is a standard BD model (see Definition 1.2.1) equipped with a (classical) probability measure on the set of states.

Definition 1.4.1 (Probabilistic BD models)**.** *A probabilistic Belnap–Dunn model is a tuple* $\mathfrak{M} = \langle W, \mu, v^+, v^- \rangle$*, such that* $\langle W, v^+, v^- \rangle$ *is a* BD *model and* $\mu : P(W) \to [0,1]$ *is a classical probability measure.*

Probabilistic models allow for lifting the (classical) probability measure on a set of states to probability on formulas of BD logic via their extensions: $\mathtt{p}^+_\mu(\varphi) = \mu(|\varphi|^+)$, $\mathtt{p}^-_\mu(\varphi) = \mu(|\varphi|^-)$. As $\mathtt{p}^+_\mu$ and $\mathtt{p}^-_\mu$ are related: $\mathtt{p}^-_\mu(\varphi) = \mu(|\varphi|^-) = \mu(|\neg\varphi|^+) = \mathtt{p}^+_\mu(\neg\varphi)$, it is sufficient to work only with $\mathtt{p}^+_\mu$, whence, the index can be omitted. It is shown in [KMR21, Lemma 1] that the function $\mathtt{p}_\mu$ satisfies properties (i)–(iii) below. Moreover, for each function $\mathtt{p}$ on the formulas of BD logic satisfying (i)–(iii) there is a probabilistic model $\langle W, \mu, v^+, v^- \rangle$ such that $\mathtt{p}(\varphi) = \mu(|\varphi|)$(see [KMR21, Theorem 4] . This allows us to take (i)–(iii) to be an axiomatisation of probability functions over Belnap–Dunn logic, which are in [KMR21] called non-standard probabilities.

Definition 1.4.2 (Non-standard probability)**.** *A map* $\mathtt{p} : \mathscr{L}_{\mathsf{BD}} \to \mathbb{R}$ *is a* non-standard probability *if it satisfies the following conditions:*

(i) $0 \leq \mathtt{p}(\varphi) \leq 1$ *(normalisation);*

(ii) if $\varphi \vdash_{\mathsf{BD}} \psi$*, then* $\mathtt{p}(\varphi) \leq \mathtt{p}(\psi)$ *(monotonicity);*

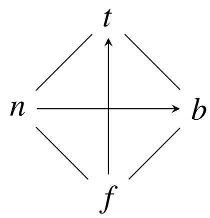


Figure 1.2: Belnap–Dunn square

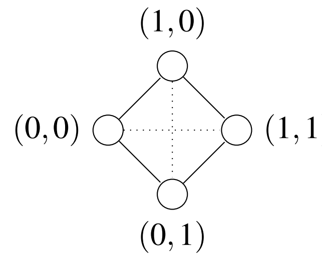


Figure 1.3: Continuous version of Belnap–Dunn square

(iii) $\mathtt{p}(\varphi \vee \psi) = \mathtt{p}(\varphi) + \mathtt{p}(\psi) - \mathtt{p}(\varphi \wedge \psi)$ *(inclusion/exclusion).*

These axioms are weaker than Kolomogorovian ones, in particular, axiom (iii) can be derived from Kolmogorovian axioms of additivity and normalization, but additivity is strictly stronger and cannot be derived from (i)–(iii) [6]. In addition, the resulting framework behaves non-classically: as $\mathtt{p}(\neg\varphi) \neq 1 - \mathtt{p}(\varphi)$ in general, this account of probability admits positive probability of classical contradictions and thus allows for a non-trivial treatment of classically inconsistent information. When defined as above, the positive and negative probabilities are mutually definable via negation as $\mathtt{p}^-(\varphi) = \mathtt{p}^+(\neg\varphi)$, and the probability of $\varphi \wedge \neg\varphi$ might be strictly bigger than zero – probabilistic information might be incomplete and/or inconsistent analogously to the background system of BD logic. We can diagrammatically represent non-standard probabilities on a continuous extension of Belnap-Dunn square (Figure 1.2), which we can see as a product bilattice $\mathcal{L}_{[0,1]} \odot \mathcal{L}_{[0,1]}$(Figure 1.3). In fact, we can see each point in the square as an ordered couple representing positive and negative probabilistic support assigned to a particular proposition. Some parts of the square suggest a natural intuitive interpretation. For example, the point $(0,0)$ corresponds to no information being provided (neither φ nor $\neg\varphi$ is supported by any state with positive measure in the underlying model), while $(1,1)$ is the point of maximally conflicting information (both φ and $\neg\varphi$ are “certain” - supported by every state with positive measure). The vertical line corresponds to ‘classical’ case in the sense that positive and negative probability of a proposition sum up to 1. The left triangle represents the area of incomplete information, while the right triangle the area of inconsistent information.

[6]Considering just the inequality $\mathtt{p}(\varphi \vee \psi) \geq \mathtt{p}(\varphi) + \mathtt{p}(\psi) - \mathtt{p}(\varphi \wedge \psi)$ in place of A3, we obtain belief functions on (finite) distributive lattices [Hal17; Zho13].

The horizontal line encodes the situation when there is equal amount of positive and negative support of the proposition, on the other words, situations where we have as much information supporting φ as contradicting it.

Example 1.4.1 (Consulting a panel)**.** *The company assembles a panel to which they ask whether a word describes the car well or not. That is, they ask how much they agree with the statements: "the car has property φ (e.g. being a family car)?" and "the car does not have property φ?" If humans were classical agents, every person would answer with a probability $\mathtt{p}$ that belongs to the vertical line of the probabilistic extension of the Belnap-Dunn square. However, experience has shown that often people don't reason classically [ASV16]. When a person answers $(\mathtt{p}^+(\varphi), \mathtt{p}^-(\varphi))$, if $\mathtt{p}^+(\varphi) + \mathtt{p}^-(\varphi) > 1$, then she is conflicted about whether the property φ describes the car, if $\mathtt{p}^+(\varphi) + \mathtt{p}^-(\varphi) < 1$, then there might be some uncertainty on how to judge whether the car has property φ.*

In this manuscript, we do not differentiate between probabilities on BD formulas over a given set of atomic formulas $\mathtt{Prop}$ and probabilities on the associated Lindenbaum algebra, that is, on the free De Morgan algebra generated by $\mathtt{Prop}$. The following Lemma ensures that those two notions indeed coincide.

Lemma 1.4.1. *There is a one-one correspondence between the functions on $\mathscr{L}_{\mathsf{BD}}$ satisfying the properties (i) - (iii) of Definition 1.4.2 and the functions on the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}$ with the same properties.*

Proof. First, we need to show that, if g is a function on $\mathscr{L}_{\mathsf{BD}}$ satisfying the properties

1. $0 \leq g(\varphi) \leq 1$ (normalisation)

2. if $\varphi \vdash_{\mathsf{BD}} \psi$, then $g(\varphi) \leq g(\psi)$ (monotonicity),

3. $g(\varphi \vee \psi) = g(\varphi) + g(\psi) - g(\varphi \wedge \psi)$ (inclusion/exclusion),

then there is a corresponding function g' on the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}$ with the same properties.

Let $g : \mathscr{L}_{\mathsf{BD}} \to \mathbb{R}$. We define $g' : \mathcal{L}_{\mathsf{BD}} \to \mathbb{R}$ as follows: $g'([\varphi]) := g(\varphi)$ where $[\varphi]$ represents the equivalence class. Notice that the monotonicity of g implies that $g(\varphi) = g(\psi)$ for any two equivalent formulas φ and ψ. Therefore, g' is well-defined. g' satisfies the three properties above because: (i) $0 \leq g'([\varphi]) \leq 1$ since $0 \leq g(\varphi) \leq 1$; (ii) if $[\varphi] \leq [\psi]$, then $\varphi \vdash_{\mathsf{BD}} \psi$, so $g(\varphi) \leq g(\psi)$, hence $g'([\varphi]) \leq g'([\psi])$; (iii) we have

$$
\begin{aligned}
g'([\varphi] \vee [\psi]) &= g'([\varphi \vee \psi]) = g(\varphi \vee \psi) && \text{(by definition of } g'\text{)} \\
&= g(\varphi) + g(\psi) - g(\varphi \wedge \psi) && \text{(by assumption)} \\
&= g'([\varphi]) + g'([\psi]) - g'([\varphi \wedge \psi]) && \text{(by definition of } g'\text{)} \\
&= g'([\varphi]) + g'([\psi]) - g'([\varphi] \wedge [\psi])
\end{aligned}
$$

The proof of the converse is similar. □

The following theorem [KMR21, Theorem 4] shows that the axioms of non-standard probability are complete with respect to probabilistic BD models.

Theorem 1.4.1 (Completeness of non-standard probabilities)**.** *Let* `Prop` *be a finite set of variables, and* $\mathtt{p}$ *a function satisfying the axioms in Definition 1.4.2. There is a probabilistic model* $\mathfrak{M} = \langle W, \mu, v^+, v^- \rangle$*, such that* $\mathtt{p} = \mathtt{p}_\mu$ *in the sense that* $\mathtt{p}(\varphi) = \mu(|\varphi|^+)$.

1.5 Related works

In the remaining of the thesis we build on the definitions and results presented here to develop a formal framework to reason with incomplete and contradictory evidence. We use Belnap–Dunn logic as our logic to model the evidence available and Łukasiewicz logic to formalise reasoning with probabilities. After [KMR21] developed a theory of probabilities over Belnap–Dunn logic, we study generalising this theory to belief functions. The theory of belief functions was first developed over Boolean algebras [Sha76; Dem67]. Recently, many results have been proposed on belief functions over non-Boolean structures. The first investigation on general lattices was started by [Bar00]. Then Grabich in [Gra09], investigates the other aspects of belief functions and Dempster's combination rule on general lattices and De Morgan algebras instead of Boolean algebras. Then Zhou in [Zho13; Zho16], tries to develop more aspects of Dempster-Shafer theory on dealt with belief functions on two particular classes of distributive lattices: bilattices and de Morgan lattices, which are actually mathematical objects in reasoning under incomplete and inconsistent information. He does his investigation based on the Birkhoff's representation theorem and then as an application he studies Dempster-Shafer theory on non-classical formalisms, such as BD-logic. There are other works in the literature, such as [BC16; CD19], in which they integrate belief functions and non-classical formalisms, to be able to take care of the limitation of the information, using belief functions and to take care of the imprecision, uncertainty or inconsistency in the knowledge-base due to imperfect data thanks to the non-classical formalisms. In addition, Frittella et al, in [Fri+20a] using the formal concept analysis, study belief function on finite lattices. We complement this related works section by stating them in each related section which are as follows: Sections 2.3, 3.1, 3.3.2, 4.1 and 4.4.

The reader can notice that we proved some lemmas in the preliminaries about the behaviour of belief and plausibility functions over De Morgan algebras. Those results might be folklore, but we could not find their mention in the literature. In what follows, we study the mathematical properties of belief and plausibility functions within the framework of Belnap–Dunn logic, and we discuss their interpretation in depth.

CHAPTER 2

Reasoning with inconsistent information

This is the very first step of the research in which we tried to propose a finer analysis of how belief can be based on information, where the confirmation comes from multiple possibly conflicting sources and is of a probabilistic nature. We use Belnap-Dunn logic and its probabilistic extensions to account for potentially contradictory information on which belief is grounded. We combine it with an extension of Łukasiewicz logic, or a bilattice logic, within a two-layer modal logical framework to account for belief. There we proposed two-layer modal logics of belief of a single agent, belief that is grounded on probabilistic information provided (positive and negative information independently) by multiple sources. The underlying logic of facts or events is chosen to be BD, the upper logic varies between BD and logics derived from Łukasiewicz logic and based on product or bilattice algebras, to systematically account for positive and negative information independently (and thus incompleteness and conflict) on both levels.

Structure of the chapter This chapter consists of three sections. In section 2.1, we give two examples to motivate our work and discuss some aggregation methods that we can use in our framework. Then in Section 2.2, we formalise the framework using a two-layered logic.

In Section 2.3, we give the conclusion and a link to the other chapters.

2.1 Case studies

Here we give some motivational examples for the project. In many scenarios we can adapt *aggregation strategies* that have been introduced on classical probabilities: a company that has access to a huge amount of heterogeneous data from various sources and uses software capable of analyzing these data. In this case it makes sense to consider aggregation methods that require a substantial computational power. A natural strategy here is to evaluate sources with respect to their reliability and aggregate them by taking their weighted average. Another kind of agent is an investigator of a criminal case who builds her opinion on the guilt of a suspect based on different pieces of evidence. We assume that all the sources are equally reliable and the investigator is very cautious and does not want to draw conclusions hastily. Hence, she relies on statements as little as all her sources agree on them, and the aggregation she uses returns the minimum of the positive and the minimum of the negative probabilities provided by the sources. If on the other hand the investigator considers all the sources being perfectly reliable, she accepts every piece of evidence and builds her belief using the aggregation maximazing both probabilities. We look at an agent who considers a set of issues, has access to (multiple) sources providing positive and negative information on the issues in form of non-standard probabilities, and builds beliefs based on information aggregated from these sources. From plethora of possible scenarios we single out two case studies that we use to illustrate the different concepts at stake.

Example 2.1.1 (Aggregating heterogeneous data)**.** *A company launching a new car model needs to decide its selling price and its advertising strategy. Hence, its data analysts must study the reports on the sells of the previous products launched by the company and the success or failure of the advertisement campaigns. This study relies on factual information such as "during the year 2015, the company sold n items of product Y", but also on statement based on statistical analyses such as "the advertisement broadcasted in June 2016 increased the sells of product Y among the 20-30 years old of 30%". The second statement is based on aggregated* *information about the buyers that might be partial and partly false. Plus, the company has access to statistical studies about the population on increase or decrease in expenses for cars.*

Example 2.1.2 (How to lead an investigation)**.** *An investigator needs to know if one of the suspects was present at the crime scene. She collects information from various sources: CCTV camera recordings, ATM logs, witnesses' statements, etc. No information of this kind is absolutely precise, and typically different sources of information contradict each other. Sources provide information of a probabilistic nature: camera recordings are imprecise due to light conditions, witnesses are not absolutely sure what they have seen. Moreover, the pieces of evidence confirming investigator's hypothesis that the suspect was present at the place of crime (that is, the positive information) are different from, and somewhat independent of, those rejecting it (that is, the negative information): there is a CCTV camera closed to the crime scene vs. ATM in a supermarket in a different city. For example, a lack of evidence supporting the suspect was present at the crime scene does not yield a proof she was not there. In the end, the investigator has to aggregate the available information and form beliefs about what likely happened.*

In many scenarios we can adapt *aggregation strategies* that have been introduced on classical probabilities. For instance, a company that has access to a huge amount of heterogeneous data from various sources and uses software capable of analyzing these data. In this case it makes sense to consider aggregation methods that require a substantial computational power. A natural strategy here is to evaluate sources with respect to their reliability and aggregate them by taking their weighted average. Another kind of agent is an investigator of a criminal case who builds her opinion on the guilt of a suspect based on different pieces of evidence. We assume that all the sources are equally reliable and the investigator is very cautious and does not want to draw conclusions hastily. Hence, she relies on statements as little as all her sources agree on them, and the aggregation she uses returns the minimum of the positive and the minimum of the negative probabilities provided by the sources. If on the other hand the investigator considers all the sources being perfectly reliable, she accepts every piece of evidence and builds her belief using the aggregation maximazing both probabilities. In what follows, we will propose two-layer modal logics of belief of a single agent, belief that is grounded on probabilistic information provided (positive and negative information independently) by multiple sources. The underlying logic of facts or events is chosen to be BD, the upper logic varies between BD and logics derived from Łukasiewicz logic and based on product or bilattice algebras, to systematically account for positive and negative information independently (and thus incompleteness and conflict) on both levels.

Aggregating probabilities

We model an agent that considers a set of topics listed by the atomic variables in $\mathtt{Prop}$, has access to sources giving information within the framework of non-standard probabilities (which we will call simply probabilities) and builds beliefs based on these sources using a so-called *aggregation strategy*. We focus on cases where the agent has no prior beliefs about the topics at stake. Depending on the context, the aggregation strategy should satisfy different properties.

A *source* s is a probability assignment over the set of formulas $s: \mathscr{L}_{\mathsf{BD}} \to [0,1] \times [0,1]$. In particular, we will later identify a source with a mass function on the BD states of a double-valuation model. An *aggregation strategy* $\mathtt{Agg}$ is a function that takes in input a set of sources $\mathcal{S} = \{s_i\}_{i \in I}$ and returns a map $\mathtt{Agg}_{\mathcal{S}}: \mathscr{L}_{\mathsf{BD}} \to [0,1] \times [0,1]$. For every $\varphi \in \mathscr{L}_{\mathsf{BD}}$, we denote $\mathtt{Agg}_{\mathcal{S}}(\varphi)^+$ (resp. $\mathtt{Agg}_{\mathcal{S}}(\varphi)^-$) the positive (resp. negative) support assigned to φ.

$\mathtt{Agg}$ is *monotone* if $\varphi \vdash \psi$ implies $\mathtt{Agg}_{\mathcal{S}}(\varphi) \leq \mathtt{Agg}_{\mathcal{S}}(\psi)$ for all $\varphi, \psi \in \mathscr{L}_{\mathsf{BD}}$ and for every $\mathcal{S}$. $\mathtt{Agg}$ is $\neg$*-compatible* if $\mathtt{Agg}_{\mathcal{S}}(\varphi)^- = \mathtt{Agg}_{\mathcal{S}}(\neg\varphi)^+$ for every $\varphi \in \mathscr{L}_{\mathsf{BD}}$ and for every $\mathcal{S}$. $\mathtt{Agg}$ *preserves probabilities* if $\mathtt{Agg}_{\mathcal{S}}$ is a probability for every $\mathcal{S}$.

Many aggregation strategies have been introduced on classical probabilities. Some of them, such as the (weighted) average, straightforwardly generalize to non-standard probabilities. In the following, we present aggregation strategies for our two case studies.

Weighted average. In Example 2.1.1, a company has access to a huge amount of heterogeneous data from various sources and to softwares to analyse these data. A natural proposal is to grade every source s_i with respect to its reliability w_i and to take the *weighted average* of the probabilities. The aggregation is then the map $\mathrm{WA} : \mathscr{L}_{\mathsf{BD}} \to [0,1] \times [0,1]$ such that, for every $\varphi \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathrm{WA}^{+}(\varphi) := \frac{\sum_{1\leq i\leq n} w_i \cdot \mathrm{p}_i^{+}(\varphi)}{\sum_{1\leq i\leq n} w_i} \qquad \text{and} \qquad \mathrm{WA}^{-}(\varphi) := \frac{\sum_{1\leq i\leq n} w_i \cdot \mathrm{p}_i^{-}(\varphi)}{\sum_{1\leq i\leq n} w_i}.$$

One can easily prove that WA preserves probabilities, it is monotone, and $\neg$-compatible. This aggregation strategy is however not feasible when modelling human reasoning.

A very cautious investigator. In Example 2.1.2, an investigator builds her opinion on the suspect based on different sources. We assume all the sources are equally reliable and the investigator does not want to draw conclusions hastily. Hence, she relies only on statements all her sources agree on. The aggregation is then the map $\mathrm{Min} : \mathscr{L}_{\mathsf{BD}} \to [0,1]\times[0,1]$ such that

$$\mathrm{Min}(\varphi) := \sqcap_{1\leq i\leq n} \mathrm{p}_i(\varphi) = \left(\min_{1\leq i\leq n} \mathrm{p}_i^{+}(\varphi), \min_{1\leq i\leq n} \mathrm{p}_i^{-}(\varphi)\right).$$

Reasoning with trusted sources. Staying in Example 2.1.2, we now assume all the sources are perfectly reliable. Hence, the investigator builds her belief on every statement supported by at least one source. The aggregation is then the map $\mathrm{Max} : \mathscr{L}_{\mathsf{BD}} \to [0,1]\times[0,1]$ such that

$$\mathrm{Max}(\varphi) := \sqcup_{1\leq i\leq n} \mathrm{p}_i(\varphi) = \left(\max_{1\leq i\leq n} \mathrm{p}_i^{+}(\varphi), \max_{1\leq i\leq n} \mathrm{p}_i^{-}(\varphi)\right).$$

Here, one has high chances of reaching contradiction. In a scientific analyses, if one gets experiments or information that is contradictory, there are two options. Either the information is incorrect or there is a mistake in the interpretation of the data. Here, if the sources are 100% reliable, reaching a contradiction state will simply indicate to our investigator that there is a flaw in her analysis of the problem and she needs to change perspective to resolve the conflict.

The two latter aggregation strategies are monotone and $\neg$-compatible, and they in general do not preserve probabilities.

2.2 Two-layer logics

To make a clear distinction between the level of events or facts, information on which the agent bases her beliefs, and the level of reasoning about her beliefs, we use a *two-layer* logical framework. The formalism originated with [FHM90; Háj98], and was further developed in [CN14; BCN20b] into an abstract algebraic framework with a general theory of syntax, semantics and completeness (we will employ this framework to derive completeness of the logics we define).

Syntax $\langle \mathscr{L}_e, \mathscr{M}, \mathscr{L}_u \rangle$ of a two layer logic $\mathscr{L}$ consists of a lower language $\mathscr{L}_e$ of events or facts (we denote formulas of $\mathscr{L}_e$ by $\varphi, \psi, \ldots$), an upper language $\mathscr{L}_u$ (we denote formulas of $\mathscr{L}_u$ by $\alpha, \beta, \ldots$), and a set of unary modalities $\mathscr{M}$ which can only be applied to a non-modal formula of $\mathscr{L}_e$, forming a modal atomic formula of $\mathscr{L}_u$ (in particular, no nesting of modalities can occur).

Semantics of a two layer logic $\mathscr{L}$ is, in the abstract approach of [CN14], based on *frames* of the form $F = \langle W, E, U, \langle \mu^{\heartsuit} \rangle_{\heartsuit \in \mathscr{M}} \rangle$, where W is a set of states, E is a local algebra of evaluation of the lower language $\mathscr{L}_e$ within the states[1], U is an upper-level algebra interpreting the modal formulas, and for each modality its semantics is given by a map $\mu^{\heartsuit} : \prod_{s \in W} E \to U$, returning a value in the upper-level algebra for a tuple of values from the lower algebra (assigned to an argument formula within the states). We write algebras, but often we need to use matrices to evaluate formulas, i.e. algebras with a set of designated values. Such a frame is called E-based and U-measured. A model is a frame equipped with valuations of $\mathscr{L}_e$ in E (the values of atomic modal formulas are then computed by μ, and values of modal formulas are computed in U in an expected way). A non-modal formula φ is valid in a model iff it is assigned a designated value in E by all the states, a modal formula α is valid in a model iff its value is designated in U. A consequence relation is defined via preserving validity in every model. It is of the sorted form $\Psi, \Gamma \vDash \xi$ where $\Psi \subseteq \mathscr{L}_e, \Gamma \subseteq \mathscr{L}_u, \xi \in \mathscr{L}_e \cup \mathscr{L}_u$.

The resulting logic as an axiomatic system $L = \langle L_e, M, L_u \rangle$ consists of an axiomatics of the lower logic L_e, modal rules (i.e. rules with non-modal premises and modal conclusion) and modal axioms (modal rules with zero premises) M, and an axiomatics of the upper logic L_u. Proofs can be defined in the expected way. We can see that $\Psi, \Gamma \vdash \varphi$ iff $\Psi \vdash_{L_e} \varphi$, and $\Psi, \Gamma \vdash \alpha$ iff $\Psi_{MR}, \Gamma \vdash_{L_u} \alpha$, where Ψ_{MR} consists of conclusions of modal rules whose premises are derivable from Ψ in L_e (for more detail see [BCN20b, Proposition 3]).

[1] For this paper, we always consider the lower algebras be the same for all states. But different algebras can be later used when modelling heterogeneous information.

2.2.1 Logic of probabilistic belief

In scenarios like that of Example 2.1.1, it is reasonable to represent agents beliefs as probabilities. In such two-layer logics, the bottom layer is that of events or facts, represented by BD-information states. A source provides probabilistic information given as a mass function on the states, multiple sources are to be aggregated with an aggregation strategy preserving probabilities. The modality is that of probabilistic belief. For the upper-layer – the logic of thus formed beliefs – we propose two logics derived from Łukasiewicz logic. The main reason to choose Łukasiewicz logic as a starting point is that it can express the probability axioms, and contains a well-behaved (continuous) implication. We however also aim at a formalism that allows us to separate the positive and negative dimensions of information or support also on the level of beliefs (just like BD does on the lower level). This motivates the use of product or bilattice algebras (those of Examples 1.1.3 and 1.1.4) on the upper level.

I. An extension of Łukasiewicz logic with bilattice negation. Consider the product of the standard algebra of Łukasiewicz logic $[0,1]_{Ł} = ([0,1], \wedge, \vee, \&_{Ł}, \rightarrow_{Ł})$ with the algebra $[0,1]^{op}_{Ł} = ([0,1]^{op}, \vee, \wedge, \oplus_{Ł}, \ominus_{Ł})$, as introduced in Example 1.1.3(3.), with only $(1,0)$ as the designated value. The logic of this product algebra (understood as the set of theorems - formulas always evaluated at $(1,0)$ - or as a consequence relation preserving the value $(1,0)$) is Łukasiewicz logic Ł. It can be axiomatized in the (complete) language $\{\rightarrow, \sim\}$ by axioms of weakening, suffixing, commutativity of disjunction, and contraposition, and the rule of Modus Ponens (see the axioms below). To be able to operate the pairs of values as a positive and negative support of formulas, we extend the signature of the algebra with the bilattice negation $\neg(a_1,a_2) = (a_2,a_1)$, and extend the language to $\{\rightarrow, \sim, \neg\}$ (notice in particular, that $\oplus$ and $\ominus$ can be defined as in Example 1.1.3). We obtain the following axioms and rules, and denote the resulting consequence relation $\vdash_{Ł(\neg)}$:

$$\alpha \to (\beta \to \alpha) \qquad \neg\neg\alpha \leftrightarrow \alpha$$
$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)) \qquad \neg{\sim}\alpha \leftrightarrow {\sim}\neg\alpha$$
$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha) \qquad ({\sim}\neg\alpha \to {\sim}\neg\beta) \leftrightarrow {\sim}\neg(\alpha \to \beta)$$
$$({\sim}\beta \to {\sim}\alpha) \to (\alpha \to \beta) \qquad \alpha, \alpha \to \beta / \beta \quad \alpha / {\sim}\neg\alpha$$

The $\neg$ negations can be pushed to the atomic formulas, and we can thus consider formulas up to provable equivalence in a *negation normal form (nnf)*, i.e. formulas built using $\{\to, {\sim}\}$ from *literals* of the form $p, \neg p$. It is easy to see, because we have $\neg{\sim}\alpha \leftrightarrow {\sim}\neg\alpha$ and $\neg(\alpha \to \beta) \leftrightarrow {\sim}{\sim}\neg(\alpha \to \beta) \leftrightarrow {\sim}({\sim}\neg\alpha \to {\sim}\neg\beta)$ provable. A procedure can be defined which turns each α into $\alpha^{\neg}$ in nnf, so that we can prove, by induction, that $({\sim}\alpha)^{\neg} \leftrightarrow {\sim}\alpha^{\neg}$ and $(\alpha \to \beta)^{\neg} \leftrightarrow {\sim}({\sim}\alpha^{\neg} \to {\sim}\beta^{\neg})$. We denote $\boxdot\Gamma := {\sim}\neg\Gamma \cup \Gamma$.

Lemma 2.2.1. *For any* finite *set of formulas* Γ, α *in a nnf,*

$$\Gamma \vdash_{Ł(\neg)} \alpha \;\; \textit{iff} \quad \textit{for some finite } \Delta : \; \boxdot\Gamma, \Delta \vdash_{Ł} \alpha,$$

where Δ *contains instances of* $\neg$*-axioms.*

Proof. The right-left direction is almost trivial: Ł is a subsystem of Ł($\neg$), and all the axioms in Δ are provable in Ł($\neg$), and, thanks to the ${\sim}\neg$-rule, $\Gamma \vdash_{Ł(\neg)} \boxdot\gamma$ for each $\gamma \in \Gamma$.

For the other direction, we proceed in a few steps. First, we denote by $\vdash_{Ł(\neg)^-}$ provability in Ł($\neg$) *without* the ${\sim}\neg$-rule. By routine induction on proofs (and using that ${\sim}\neg$ distributes from/to implications and negations), we can see that

$$\Gamma \vdash_{Ł(\neg)} \alpha \;\; \text{iff} \quad \boxdot\Gamma \vdash_{Ł(\neg)^-} \alpha.$$

Then we can list all the instances of $\neg$-axioms in the proof in Δ, and obtain:

$$\boxdot\Gamma \vdash_{Ł(\neg)^-} \alpha \;\; \text{iff} \quad \boxdot\Gamma, \Delta \vdash_{Ł} \alpha.$$

First, note that we can list in Δ all instances of $\neg$-axioms for all subformulas of Γ, α as well and still keep the Lemma valid. This will come handy in the following proof. Second, we stress that in the final proof $\boxdot\Gamma, \Delta \vdash_{Ł} \alpha$ in Ł, we still use language of Ł($\neg$), where formulas starting with $\neg$ are seen from the point of view of Ł as atomic. □ □

Lemma 2.2.1 provides a translation of provability in Ł($\neg$) to provability in Ł and allows us to observe that the extension of Ł by $\neg$ is conservative. Now, using finite completeness of Ł, we can see that Ł($\neg$) is finitely strongly complete w.r.t. $[0,1]_{Ł} \times [0,1]_{Ł}^{op}$:

Lemma 2.2.2 (Finite strong standard completeness of Ł($\neg$))**.** *For a* finite *set of formulas* Γ*,*

$$\Gamma \vdash_{Ł(\neg)} \alpha \ \textit{ iff } \quad \forall e : \mathscr{L} \to [0,1]_{Ł} \times [0,1]_{Ł}^{op}\ (e[\Gamma] \subseteq \{(1,0)\} \to e(\alpha) = (1,0)).$$

Proof. The left-right direction is soundness, and consists of checking that the axioms are valid and the rules sound. We only do some cases:

First the $\sim\neg$-rule: assume that e is given and $e(\alpha) = (1,0)$. Then $e(\sim\neg\alpha) = \sim\neg(1,0) = \sim(0,1) = (1,0)$.

Next, for any e, $e(\sim\neg(\alpha \to \beta)) = \sim\neg(e(\alpha) \to e(\beta)) = \sim\neg(e(\alpha)_1 \to_{Ł} e(\beta)_1, \sim_{Ł}(e(\beta)_2 \to_{Ł} e(\alpha)_2)) = ((e(\beta)_2 \to_{Ł} e(\alpha)_2), \sim_{Ł}(e(\alpha)_1 \to_{Ł} e(\beta)_1))$,

and, $e(\sim\neg\alpha \to \sim\neg\beta) = \sim\neg e(\alpha) \to \sim\neg e(\beta) = (\sim_{Ł}e(\alpha)_2, \sim_{Ł}e(\alpha)_1) \to (\sim_{Ł}e(\beta)_2, \sim_{Ł}e(\beta)_1) = (\sim_{Ł}e(\alpha)_2 \to_{Ł} \sim_{Ł}e(\beta)_2, e(\alpha)_1 \&_{Ł} \sim_{Ł}e(\beta)_1) = ((e(\beta)_2 \to_{Ł} e(\alpha)_2), \sim_{Ł}(e(\alpha)_1 \to_{Ł} e(\beta)_1))$.

Last, for any e, $e(\neg\sim\alpha) = \neg\sim e(\alpha) = (\sim_{Ł}e(\alpha)_2, \sim_{Ł}e(\alpha)_1) = \sim\neg e(\alpha) = e(\sim\neg\alpha)$.

For the other direction, let us assume that $\Gamma \nvdash_{Ł(\neg)} \alpha$. Then for some finite Δ containing instances of $\neg$-axioms (in particular those for subformulas of Γ, α), we have $\boxdot\Gamma, \Delta \nvdash_{Ł} \alpha$. Because Ł is finitely standard complete, there is an evaluation $e : \mathrm{At} \to [0,1]_{Ł}$ sending all formulas in $\boxdot\Gamma, \Delta$ to 1, while $e(\alpha) < 1$. Here, At contains literals from Γ, α of the form $p, \neg p$, and atoms and formulas of the form $\neg\delta$ from $\boxdot\Gamma, \Delta$. We define $e' : \mathtt{Prop} \to [0,1]_{Ł} \times [0,1]_{Ł}^{op}$ by $e'(p) = (e(p), e(\neg p))$. We can then prove, by routine induction, that for each formula $e'(\beta) = (e(\beta), e(\beta^{\neg}))$. We use the fact that $e[\Delta] \subseteq \{1\}$, and Δ contains all instances of $\neg$-axioms for all subformulas of Γ, α.

We now immediately see that $e'(\alpha) < (1,0)$, because $e(\alpha) < 1$. To prove that indeed $e'[\Gamma] \subseteq \{(1,0)\}$, we use the fact that $e[\boxdot\Gamma] \subseteq \{1\}$: as for all $\gamma \in \Gamma$, $e(\sim\neg\gamma) = 1$, $e(\neg\gamma) = e(\gamma^{\neg}) = 0$. For the latter, we again need to use the fact that $e[\Delta] \subseteq \{1\}$, and Δ contains all instances of $\neg$-axioms for all subformulas of Γ, as they prove, by means of Ł, that $\neg\gamma \leftrightarrow \gamma^{\neg}$, and e has to respect that. Now we conclude, that for all $\gamma \in \Gamma$, $e'(\gamma) = (e(\gamma), e(\gamma^{\neg})) = (1,0)$.
□ □

We can now put together the two-layer syntax of the first two-layer logic to be

- $\mathscr{L}_e = \{\wedge, \vee, \neg\}$ language of BD,
- $\mathscr{M} = \{B\}$ a belief modality,
- $\mathscr{L}_u = \{\to, \sim, \neg\}$ language of Ł$(\neg)$.

The intended models can be described as follows: the lower layer is a double-valuation model of BD $(W, \Vdash^+, \Vdash^-)$ (a set of states W, and the two support relations, which in fact can be seen as arising from an evaluation of formulas of BD locally in the states in the product bilattice $2 \odot 2$, which is isomorphic to $\mathbf{4}$, as noted in Subsection 1.2). A source is given by a mass function on the states $\mathtt{m}_i : W \to [0,1]$, we assume there are n sources, and each source comes with a weight $w_i \in [0,1]$. For a non-modal formula $\varphi \in \mathscr{L}_e$, we obtain the value $||B\varphi|| \in [0,1]_{Ł} \times [0,1]_{Ł}^{op}$ as a pair of its positive and negative probabilities as follows. First, for each source $\mathtt{m}_i$, we have $(\sum_{v \Vdash^+ \varphi} \mathtt{m}_i(v), \sum_{v \Vdash^- \varphi} \mathtt{m}_i(v)) = (\mathtt{p}_i^+(\varphi), \mathtt{p}_i^-(\varphi))$. Now, applying the weighted average aggregation strategy we obtain

$$||B\varphi|| = \left(\frac{\sum_{1 \le i \le n} w_i \cdot \mathtt{p}_i^+(\varphi)}{\sum_{1 \le i \le n} w_i}, \frac{\sum_{1 \le i \le n} w_i \cdot \mathtt{p}_i^-(\varphi)}{\sum_{1 \le i \le n} w_i} \right).$$

The modal part M of the two-layer logic consists of two axioms and a rule reflecting directly the axioms of probabilities listed in Subsection 1.4:[2]

$$B(\varphi \vee \psi) \leftrightarrow (B\varphi \ominus B(\varphi \wedge \psi)) \oplus B\psi \qquad B\neg\varphi \leftrightarrow \neg B\varphi$$
$$\varphi \vdash_{\mathsf{BD}} \psi / \vdash_{Ł(\neg)} B\varphi \to B\psi$$

The resulting logic is $(\mathsf{BD}, M, Ł(\neg))$. As BD is locally finite and strongly complete w.r.t. $\mathbf{4}$, and Ł$(\neg)$ is finitely strongly complete w.r.t. $[0,1]_{Ł} \times [0,1]_{Ł}^{op}$, we can apply [CN14, Theorems 1 and 2] directly to obtain finite strong completeness (soundness of the modal axioms and rules is easy to see). But first, we need to observe that the frames as we have described them can be seen within the framework of [CN14]:

The frames, seen in the format of [CN14], are $F = (W, \mathbf{4}, [0,1]_{Ł} \times [0,1]_{Ł}^{op}, \mu^B)$, formulas of $\mathscr{L}_e$ are evaluated locally in the states of W using $\mathbf{4}$, as in the four-valued models for BD (which

[2] Considering just the right-left implication in the first axiom, we can express belief functions.

we can see as equivalent to the double-valuation models). The interpretation of modalities μ^B is computed as follows. A source is given by a mass function on the states $\mathtt{m}: W \to [0,1]$. Each source comes with a weight $w_i \in [0,1]$. Given $\mathbf{e} \in \prod_{v\in W} \mathbf{4}$, we obtain, for each source $\mathtt{m}_i$, first the following sums of weights over states: $(\sum_{\mathbf{e}_v\in\{t,b\}} \mathtt{m}_i(v), \sum_{\mathbf{e}_v\in\{f,b\}} \mathtt{m}_i(v)) = (\mathtt{p}_i^+(\mathbf{e}), \mathtt{p}_i^-(\mathbf{e}))$. The assignment μ^B now computes the weighted average of those as follows:

$$\mu^B(\mathbf{e}) = \mathrm{WA}(\mathbf{e}) = \left(\frac{\sum_{1\leq i\leq n} w_i \cdot \mathtt{p}_i^+(\mathbf{e})}{\sum_{1\leq i\leq n} w_i}, \frac{\sum_{1\leq i\leq n} w_i \cdot \mathtt{p}_i^-(\mathbf{e})}{\sum_{1\leq i\leq n} w_i}\right).$$

Thus, for a non-modal formula $\varphi \in \mathscr{L}_e$, applying μ^B to the tuple of its values in the states (which we denote by $||\phi||$), we obtain the value of $B\varphi$ as $||B\varphi|| \in [0,1]_Ł \times [0,1]_Ł^{op}$ as a pair of its positive and negative probabilities as follows: First, for each source we have[3]

$$\left(\sum_{v\Vdash^+\varphi} \mathtt{m}_i(v), \sum_{v\Vdash^-\varphi} \mathtt{m}_i(v)\right) = (\mathtt{p}_i^+(\varphi), \mathtt{p}_i^-(\varphi)).$$

Now, applying the weighted average aggregation we obtain

$$||B\varphi|| = \mu^B(||\phi||) = \mathrm{WA}(||\phi||) = \left(\frac{\sum_{1\leq i\leq n} w_i \cdot \mathtt{p}_i^+(\varphi)}{\sum_{1\leq i\leq n} w_i}, \frac{\sum_{1\leq i\leq n} w_i \cdot \mathtt{p}_i^-(\varphi)}{\sum_{1\leq i\leq n} w_i}\right).$$

We can now conclude the completeness as follows:

Corollary 2.2.1. $(\mathsf{BD}, M, Ł(\neg))$ *is* finitely strongly complete *w.r.t.* **4** *based,* $[0,1]_Ł \times [0,1]_Ł^{op}$-*measured frames validating M.*

In such frames, μ^B interprets B as a probability: For a frame to validate the axioms in M means they are sent to $(1,0)$, by an evaluation in $[0,1]_Ł \times [0,1]_Ł^{op}$ induced by μ^B over the lower state valuations (which determines values of modal atomic formulas). An equivalence $\alpha \leftrightarrow \beta$ is evaluated at $(1,0)$ iff the values of α and β are equal. $B(\varphi)^M = \mu^B(\varphi^M) = (\mathtt{p}^+(\varphi), \mathtt{p}^-(\varphi))$. Therefore, the first two axioms say that

$$\mathtt{p}^+(\varphi \vee \psi) = (\mathtt{p}^+(\varphi) - \mathtt{p}^+(\varphi \wedge \psi)) + \mathtt{p}^+(\psi) \text{ and } \mathtt{p}^+(\neg\varphi) = \neg\mathtt{p}^-(\varphi)$$
$$\mathtt{p}^-(\varphi \vee \psi) = (\mathtt{p}^-(\varphi) - \mathtt{p}^-(\varphi \wedge \psi)) + \mathtt{p}^-(\psi) \text{ and } \mathtt{p}^-(\neg\varphi) = \neg\mathtt{p}^+(\varphi).$$

[3]The value of φ in v being among $\{t,b\}$ means it is positively supported in v, i.e. $v \Vdash^+ \varphi$. Similarly $\{f,b\}$ means negative support.

Similarly, the fact that the frame validates the rule say that p^+ (p^-) are monotone (antitone) w.r.t. $\varphi \vdash_{\mathsf{BD}} \psi$. Analogous observation holds for the case the upper logic is the bilattice one.

From [KMR21, Theorem 4], we know that it is the induced probability function of exactly one mass function on the BD canonical model, which in fact yields completeness w.r.t. the intended frames described above (with a single source).

II. A bilattice Łukasiewicz logic. Alternatively, if we wish to use full expressivity of a bilattice language, we can take in the upper layer $\mathscr{L}_u = \{\wedge, \vee, \sqcap, \sqcup, \subset, \neg, 0\}$ to be the language of the *product residuated bilattice* $[0,1]_{Ł} \odot [0,1]_{Ł} = ([0,1] \times [0,1], \wedge, \vee, \sqcap, \sqcup, \supset, \neg, (0,0))$, defined in the spirit of [JR12] in Example 1.1.4. We evaluate formulas of the upper logic in the matrix $([0,1]_{Ł} \odot [0,1]_{Ł}, F)$ with $F = \{(1,a) \mid a \in [0,1]\}$ as the designated values, so that we send 0 to $(0,0)$. The constants and connectives $\top, \bot, 1, *, \to, \sim, \oplus, \ominus$ are definable as follows:

$$
\begin{aligned}
{\sim}\alpha &:= (\alpha \supset 0) \sqcup \neg(\neg\alpha \supset 0) & \top &:= 0 \supset 0 \;\; \bot := \neg\top \;\; 1 := {\sim}0 \\
\alpha \to \beta &:= (\alpha \supset \beta) \wedge (\neg\beta \supset \neg\alpha) & \alpha \oplus \beta &:= ({\sim}\alpha \supset \beta) \sqcup \neg({\sim}\neg\alpha \supset \neg\beta) \\
\alpha * \beta &:= \neg(\beta \to \neg\alpha) & \alpha \ominus \beta &:= {\sim}(\alpha \supset \beta) \sqcup \neg{\sim}(\neg\alpha \supset \neg\beta)
\end{aligned}
$$

For an evaluation e, it holds that $e(\alpha \to \beta) \in F$ iff $e(\alpha \to \beta) \geq_t (1,1)$ iff $e(\alpha) \leq_t e(\beta)$. The upper logic L_u as a consequence relation is defined to be

$$\Gamma \vDash_{Ł\odot Ł} \alpha \text{ iff } \forall e (e[\Gamma] \subseteq F \to e(\alpha) \in F).$$

The intended frames now use $[0,1]_{Ł} \odot [0,1]_{Ł}$ as the upper algebra, otherwise semantics of atomic modal formulas is computed (from multiple sources) as in the previous logic. We also obtain literally the same modal axioms M as above. Only here, apart from a very generic completeness w.r.t. **4** based frames, where the upper algebra is an algebra (in fact the Lindenbaum-Tarski algebra) of the upper logic, we cannot provide a better insight at the moment and leave axiomatization of L_u, and completeness w.r.t $[0,1]_{Ł} \odot [0,1]_{Ł}$-measured frames to further investigations (cf. footnote 3).

2.2.2 Logic of monotone coherent belief

The simplest logic we propose to deal with scenarios like the one of Example 2.1.2 is of the form $(\mathsf{BD}, M, \mathsf{BD})$. Both lower and upper languages are the language of BD, $\mathscr{M}$ consists of a single belief modality B. The intended frames are based on double-valuation semantics of BD as before, only now we evaluate formulas of the upper logic in the bilattice $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$ on Figure 1 (right). A source is given by a mass function on the states $m_i : W \to [0,1]$, we again assume there are n sources. For a non-modal formula φ, we obtain the value $||B\varphi|| \in \mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$ as follows. First, for each source $\mathtt{m}_i$, we have $(\sum_{v \Vdash^+ \varphi} \mathtt{m}_i(v), \sum_{v \Vdash^- \varphi} \mathtt{m}_i(v)) = (\mathtt{p}_i^+(\varphi), \mathtt{p}_i^-(\varphi))$. Now, applying the Min aggregation strategy we obtain

$$||B\varphi|| = \left(\min_{1 \leq i \leq n} \mathtt{p}_i^+(\varphi), \min_{1 \leq i \leq n} \mathtt{p}_i^-(\varphi) \right).$$

Similarly, we may use the Max aggregation strategy when reasoning with trusted sources. As before, we can see the frames inside the framework of [CN14] to derive completeness: frames are of the form $F = (W, \mathbf{4}, \mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}, \mu^B)$ where $\mu^B : \prod_{v \in W} \mathbf{4} \to \mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$ computes the Min (Max) aggregation of the probabilities given by the individual sources. In general this aggregation strategy does not yield a probability, but it is monotone and $\neg$-compatible. This motivates considering logic $(\mathsf{BD}, M, \mathsf{BD})$, where the modal part M consists of the following two axioms and a rule

$$B\neg\varphi \dashv\vdash_{\mathsf{BD}_u} \neg B\varphi \qquad \varphi \vdash_{\mathsf{BD}_e} \psi / B\varphi \vdash_{\mathsf{BD}_u} B\psi.$$

As BD is strongly complete w.r.t. both $\mathbf{4}$ and $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$[4], we can apply [CN14, Theorem 1] to conclude that $(\mathsf{BD}, M, \mathsf{BD})$ is strongly complete w.r.t. $\mathbf{4}$-based $\mathbf{L}_{[0,1]} \odot \mathbf{L}_{[0,1]}$-measured frames validating M. In such frames, μ^B interprets B as a monotone and $\neg$-compatible assignment (not necessarily a probability). We cannot in general see it as coming from a measure, or a set of measures[5], on the lower states (to recover sources), and connect it with the intended semantics.

[4]Because it has $(2 \odot 2, \{(1,0),(1,1)\})$ as a sub-matrix: the obvious embedding is a *strict* homomorphism of de Morgan matrices - it preserves and reflects the filters.

[5]It is not hard to provide an example of such assignment which cannot be obtained by Min (Max) aggregation of probabilities.

One could however replace the upper language with the full bilattice language, consider modalities indexed by sources, and express the Min (Max) aggregations explicitly using $\sqcap, \sqcup$ connectives.

2.3 Conclusion

We have proposed two-layer logics of belief based on potentially inconsistent probabilistic information coming from multiple sources. The framework keeps positive and negative aspect of information (support, evidence, belief) separate, though inter-linked, in both layers of the semantics, and thus allows for reasoning with inconsistencies, in contrast to getting rid of them. Doing so, we believe we have laid groundwork to a modular framework to model reasoning with inconsistent probabilistic information.

We see our contribution in the following: to see how Belnap-Dunn's logic *BD* (on the lower layer, and behind the non-standard probabilities) can be combined with many-valued reasoning on the upper layer provides a novel example of two-layer logics for reasoning under uncertainty. The only examples considered so far used either classical logic [FHM90], or quantitative reasoning in form of linear inequalities on the upper layer [Zho13]. The logic $Ł(\neg)$, extending Łukasiewicz logic with bi-lattice negation, we introduced in Subsection 2.2.1 and proved its finite strong standard completeness, is to our best knowledge new and might be of independent interest. (The same can be said about the bi-lattice Łukasiewicz logic, which however remains to be axiomatized and its completeness studied.)

In the next chapter, we build on [Fri+20b] that generalises Dempster-Shafer theory [Sha76] to finite lattices, to adapt the theory to the BD-based setting. This allows us to interpret Dempster-Shafer theory over BD logic.

We also would like to look at further research directions. For instance, to cover cases when a source does not give an opinion about each formula of the language, we need to account for sources providing partial probability maps. Also cases where sources provide heterogeneous information need to be included.

An important direction to move further is to capture dynamics of information and belief given by updates on the level of sources, and to generalize the framework to the multi agent setting involving group modalities and dynamics of belief. Specifically, forming group belief, like common and distributed belief, will involve communication and sharing or pooling of sources. It might call for a use of various upper-layer languages, among those we see the ones with additional (nestable) modalities inside the upper logic to account for reflected, higher-order beliefs, in contrast to the beliefs grounded directly in the sources. In Chapter 4, we make a first step in that direction by studying update of belief functions.

CHAPTER

3 Belief functions over Belnap–Dunn logic

3.1 Introduction to the chapter and related works

Many generalisations of traditional probability theory have been proposed via possibility functions [DP07; Zad82], belief functions [Sha76; SK94], lower and upper probabilities [Dem67; WF82], inner and outer measures [Hal50] credal sets [Lev83] and so on. Each proposal trying to deal with the fact that either one cannot know the exact probability measure describing the world, or this probability measure does not even exist. Most of these proposals have been done within the framework of classical logic and classical reasoning. Here, we aim at developing this very fruitful theory to the realm of non-classical logic. Some attempts already exists by generalising probabilities and belief functions over distributive lattices. The first investigation on general lattices was started by [Bar00]. Then Grabich in [Gra09], investigates the other aspects of belief functions and Dempster's combination rule on general lattices and De Morgan algebras instead of Boolean algebras. Then Zhou in [Zho13; Zho16], tries to develop more aspects of Dempster-Shafer theory on dealt with belief functions on two particular classes of distributive lattices: bilattices and de Morgan lattices, which are actually mathematical objects in reasoning under incomplete and inconsistent information. He does his investigation based on the Birkhoff's representation theorem and then as an application he studies Dempster-Shafer theory on non-classical formalisms, such as BD-logic. There are other works in the literature, such as [BC16; CD19], in which they integrate belief functions and non-classical formalisms, to be able to take care of the limitation of the information, using belief functions and to take care of the imprecision, uncertainty or inconsistency in the knowledge-base due to imperfect data thanks to the non-classical formalisms. In addition, Frittella et al, in [Fri+20a], using the formal concept analysis, study belief function on finite lattices. As we said, belief functions and Dempster-Shafer's combination rule on De Morgan algebras, in [Zho13; Zho16; Gra09], so as a result on BD logic is already studied partially. One of the things that we focus on is to study plausibility functions separately because it has a different interpretation and behavior than the classical case. One of the aspects that we consider is the interpretation and the behaviour of belief, plausibility functions and the Dempster Shafer Combination rule on BD logic. Also the framework that we present is independent from the algebraic structure. In the sense that, this model always gives us two beliefs (plausibility or whatever), one defined over the set of states which behaves totally classic and the other one which is defined over the lindenbaum algebra and because of the presence of the negation is not well-behaved. We always can apply the existing classical results on our model based using the first ones.

Structure of the Chapter: In this chapter, we focus generalising the theory of belief functions within the framework of Belnap–Dunn logic. We build on the theory of non-standard probabilities over Belnap–Dunn logic [KMR21] presented in Section 1.4. In Section 3.2, we present the proof of [KMR21, Theorem 4] where the authors introduce the canonical models for probabilistic BD logic. In Section 3.3, we discuss the interpretation of belief and mass functions in the context of evidence-based reasoning and present Dempster–Shafer combination rule. We show a well-known example in which it gives counterintuitive results within the framework of classical logic, and we discuss the added value of reasoning with belief functions within the framework of Belnap Dunn logic. In Section 3.4, we introduce DS models to interpret belief and plausibility on formulas of BD logic. Then, we present different interpretations of belief and plausibility that lead to different generalisations of the classical definition. In Section 3.5, we discuss some results building on this work.

3.2 Preliminaries on non-standard probabilities

As we mentioned before in Theorem 1.4.1, the axioms of non-standard probability are complete with respect to probabilistic BD models, that is, for a $\mathtt{p}$ a function satisfying the axioms in Definition 1.4.2. There is a probabilistic model $\mathfrak{M} = \langle W, \mu, v^+, v^- \rangle$, such that $\mathtt{p} = \mathtt{p}_\mu$ in the sense that $\mathtt{p}(\varphi) = \mu(|\varphi|^+)$. Here we need to explain and reprove this theorem in our ways. The construction of a canonical model used in the proof of this theorem uses the fact we previously mentioned that each formula of BD logic can be uniquely represented in *irredundant disjunctive normal form* (iDNF) (see Section 1.2). There is a straightforward correspondence between BD formulas in iDNF and sets of sets of literals:[1]

$$\varphi = \bigvee_i \bigwedge_j l^i_j \to \{\{l^1_1, \dots, l^1_{n_1}\}, \dots, \{l^m_1, \dots, l^m_{n_m}\}\},\ l^i_j \in \mathsf{Lit}$$

In other words, each formula corresponds to a disjunction of a family of sets of literals interpreted conjunctively.

Definition 3.2.1 (Canonical model)**.** *The canonical* BD *model is a tuple* $\mathfrak{M}_c = \langle S_c, v^+_c, v^-_c \rangle$, *where* $S_c = P(\mathsf{Lit})$ *and the valuations* $v^+_c, v^-_c : \mathtt{Prop} \to P(S_c) = P(P(\mathsf{Lit}))$ *are defined as* $v^+_c(p) = \{s \mid p \in s\}$, $v^-_c(p) = \{s \mid \neg p \in s\}$ *which are uppersets in the poset* $\langle P(\mathsf{Lit}), \subseteq \rangle$.

Proof of Theorem 1.4.1. Consider the canonical model $\mathfrak{M}_c = \langle P(\mathsf{Lit}), v^+_c, v^-_c \rangle$ introduced above. Notice that, for every $p \in \mathtt{Prop}$, $v^+_c(p)$ and $v^-_c(p)$ are uppersets[2] in the poset $\langle P(\mathsf{Lit}), \subseteq \rangle$. The positive extension of a formula φ in iDNF, $\varphi = \bigvee_{i=1}^n \gamma_i$ for some conjunctions of literals γ_i is the set $|\varphi|^+ = \{s \mid s \models \varphi\} = \{s \mid s \supseteq \mathsf{Lit}(\gamma_i)$ for some $i, 1 \leq i \leq n\}$. Thus, extensions of formulas are uppersets in the poset $\langle P(\mathsf{Lit}), \subseteq \rangle$, the sets of literals γ_i generate the upperset $|\varphi|^+$ and in fact they are the minimal set of its generators. This correspondence is one-to-one between formulas in disjunctive normal forms and uppersets in the Boolean algebra $P(\mathsf{Lit})$: each extension is an upperset, and each upperset (other than $\varnothing$ and $P(\mathsf{Lit})$)[3] is a positive extension (of the formula given in iDNF using the finite antichain of the generators of the

[1] This correspondence is not one to one, as some of the sets correspond to a formula in DNF, but not in iDNF.

[2] Indeed, for every $s, s' \in P(\mathsf{Lit})$, if $s \models^+ p$ and $s \subseteq s'$, then $s' \models^+ p$.

[3] Notice that $\varnothing$ and $P(\mathsf{Lit})$ are the extensions of $\bot$ and $\top$.

upperset). Moreover, the mapping $\varphi \mapsto |\varphi|^+$ is an isomorphism of both structures understood as distributive lattices ($|\varphi \vee \psi|^+ = |\varphi|^+ \cup |\psi|^+$ and $|\varphi \wedge \psi|^+ = |\varphi|^+ \cap |\psi|^+$), hence a non-standard probability function p on formulas defines a non-standard probability function p' on uppersets (other than $\varnothing$ and $P(\mathsf{Lit})$) as $\mathrm{p}'(|\varphi|^+) = \mathrm{p}(\varphi)$. We can extend p' to $\varnothing$ and $P(\mathsf{Lit})$ in the obvious way: $\mathrm{p}'(\varnothing) = 0$ and $\mathrm{p}'(P(\mathsf{Lit})) = 1$ and we obtain what is in [Zho13] called probability function on a distributive lattice. Then we use Lemma 3.5 in [Zho13], which says that a probability function on a distributive lattice $\mathcal{L}$ can be uniquely extended to a probability function on the Boolean algebra $B_{\mathcal{L}}$ generated by the lattice $\mathcal{L}$.

Let us note that in [KMR21] an alternative proof is provided. It uses the fact that the required probability measure μ on the canonical model is generated by its values on singletons $\{s\}, s \in P(\mathsf{Lit})$. As sets of literals are ordered by inclusion with the maximal element being the set corresponding to the conjunction of all the literals, we can define the measure on singletons inductively. We start with the conjunctive clause $\gamma_{max} = \bigwedge_{l \in \mathsf{Lit}} l, \mathsf{Lit} = \{1, \dots, n\}$ and assign $\mu(\{l \mid l \in \mathsf{Lit}\}) = \mathrm{p}(\gamma_{max})$. In the induction step for $s = \{l_1, \dots, l_k\}, k \leq n$, we define $\mu(\{s\}) = \mathrm{p}(\bigwedge_{i=1 \dots k} l_i) - \sum_{s \subset s'} \mu(\{s'\})$. □

3.3 Evidential reasoning via mass functions on algebras

The classical treatment of probability has two distinctive traits. First, the probability is assumed to be 'compositional', in the sense the probability measure of any given event is uniquely determined by the probabilities of elementary events. Second, and related to the first, is that probabilities of all events are assumed to be known (or at least, knowable). Formally, these assumptions lead to sample spaces being Boolean algebras.

It may be reasonably argued that these assumptions are too optimistic and do not correspond to the situations one encounters in practice. Indeed, given the probability assignments of some elementary events $a_1, \dots, a_m$, one may not be able to infer probabilities of their combinations if said assignments were obtained by different methods (i.e., the data were heterogeneous). On a related note, it is not necessarily the case that the probability of all elementary events is known even if one somehow obtained an assignment for a complex event composed of those.

Taking that into account, one can generalize the classical approach to probability in two (compatible) ways. First, given a Boolean algebra $\mathcal{B}$, one can define the probability assignment on its proper subalgebra $\mathcal{B}'$. The values of the events in $\mathcal{B} \setminus \mathcal{B}'$ can be then approximated via more general uncertainty measures, e.g., belief and plausibility functions or inner and outer measures. The other approach is to represent the sample space not as a Boolean algebra but in the form of another, more general structure.

Belief and plausibility functions Belief functions were introduced in [Sha76] as a generalisation of probabilities for the case where the exact compositional uncertainty measures are not given to the entire sample space of events. Originally, they were defined on Boolean algebras, however, later work [Bar00; Gra09; Zho13; Fri+20a] saw them further expanded on arbitrary and distributive lattices. In this section, we will use a combination of the two approaches given above and consider belief (and their dual counterparts, plausibility functions) on De Morgan algebras of which Boolean algebras are a particular case and which themselves are a special case of distributive lattices equipped with negation.

In the standard approach both belief and plausibility use in fact the same information represented by the mass function, but deal with it in a different way. While we can see belief as the amount of information which directly supports the statement in question, plausibility represents the amount of information which does not contradict the statement. As Halpern says: "$Plaus_m(U)$ can be thought of as the sum of the probabilities of the evidence that is compatible with the actual world being in U." ([Hal17], p. 38). This idea is captured in the definition of plausibility via mass function: $\mathtt{pl}(A) = \sum\limits_{A\cap B\neq\varnothing} \mathtt{m}(B)$. Alternatively, we can understand plausibility as a measure of the information, which does not support the negation of the hypothesis: $\mathtt{pl}(A) = 1 - \mathtt{bel}(A^c) = \sum\limits_{B \nsubseteq A^c} \mathtt{m}(B)$. We can also see belief and plausibility as approximations, as a lower and an upper bound for the 'true' probability: $\mathtt{bel}(A) \leq \mathtt{p}(A) \leq \mathtt{pl}(A)$. While in the classical case all these readings coincide, in the case of BD logic they do not, which gives us several possibilities of defining belief/plausibility pairs.

Dempster–Shafer combination rule on powerset algebras Dempster–Shafer theory [Dem68; Sha76] is a formal framework for decision-making under uncertainty in situations in which some propositions cannot be assigned probabilities. The core proposal of Dempster–Shafer theory is that, in such cases, the missing value can be replaced by a range of values, the lower and upper bounds of which are assigned by belief and plausibility functions. In fact, the correspondence between belief functions and mass functions is used to formalise probabilistic reasoning based on pieces of evidence. A mass function is assigned to each piece of evidence to encode the information contained in the evidence. For instance, if an expert states that they are 70% certain that $p \vee q$ is true and that they do not give more information, one would assign the following mass function to this piece of evidence: $\mathtt{m}(p \vee q) = 0.7$ and $\mathtt{m}(\top) = 0.3$. Here, the remaining mass is assigned to $\top$, because it represents the non-informative statement. In the classical case, mass functions, belief functions and plausibility functions are connected via the following interpretation. While mass function represents the amount of evidence committed exactly to a particular statement, we can see belief as collecting information which directly supports the statement in question, while plausibility represents the amount of information which does not contradict the statement. Belief (resp., plausibility) is given by the sum of masses of the propositions implying (resp., not contradicting) it. One can already observe that belief and plausibility are connected via the negation and the notion of contradiction. Therefore, shifting from classical logic to BD logic will impact the definition of plausibility and the connection between belief and plausibility. In fact, this will open the door to many alternative definitions of plausibility.

Since, a priori, a mass function is assigned to each piece of evidence, the natural next step is to define a way to combine the information obtained from each piece of evidence. In what follows, we discuss Dempster–Shafer combination rule and its interpretation on powerset algebras. Then, we motivate interpreting it on De Morgan algebras to model more accurately and in a more informative manner situations in which one handles contradictory evidence.

Definition 3.3.1 (Dempster–Shafer combination rule over a powerset algebra)**.** *Let* $\mathtt{m}_1$ *and* $\mathtt{m}_2$ *be two mass functions on a powerset algebra* $P(S)$*. Dempster–Shafer combination rule* *computes their aggregation* $\mathtt{m}_{1\oplus 2}$ *(or* $\mathtt{m}_1 \oplus \mathtt{m}_2$*) as follows.*

$$\begin{aligned} \mathtt{m}_{1\oplus 2} : P(S) &\to [0,1] \\ X &\mapsto \begin{cases} 0 & \text{if } X = \varnothing \\ \dfrac{\sum\{\mathtt{m}_1(X_1)\cdot\mathtt{m}_2(X_2) \mid X_1 \cap X_2 = X\}}{\sum\{\mathtt{m}_1(X_1)\cdot\mathtt{m}_2(X_2) \mid X_1 \cap X_2 \neq \varnothing\}} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1}$$

Example 3.3.1 (Two disagreeing experts. Classical reasoning)**.** *A patient is sick, and two doctors are asked their opinions about which disease the patient has. Three diseases are being considered:* $S = \{a,b,c\}$*. It is assumed that the experts are infallible, and that the patient can have one and only one of the considered diseases. Therefore, the events a, b and c are incompatible and exhaustive. Expert 1 thinks that the patient has disease a with certainty* 0.9 *and that it is very unlikely the patient has disease b, therefore assigning certainty* 0.1 *to that option. Expert 2 thinks that the patient has disease c with certainty* 0.9 *and that it is very unlikely the patient has disease b, therefore assigning certainty* 0.1 *to that option.*

The opinion of expert 1 is described by the mass function $\mathtt{m}_1 : P(\{a,b,c\}) \to [0,1]$ *such that* $\mathtt{m}_1(\{a\}) = 0.9$*,* $\mathtt{m}_1(\{b\}) = 0.1$*, and* $\mathtt{m}_1(x) = 0$ *otherwise. The opinion of expert 2 is described by the mass function* $\mathtt{m}_2 : P(\{a,b,c\}) \to [0,1]$ *such that* $\mathtt{m}_2(\{b\}) = 0.1$*,* $\mathtt{m}_2(\{c\}) = 0.9$*, and* $\mathtt{m}_2(x) = 0$ *otherwise.*

The aggregated mass function $\mathtt{m}_{1\oplus 2}$*, using Dempster–Shafer combination rule, is as follows:*

$$\mathtt{m}_{1\oplus 2}(x) = \begin{cases} 1 & \text{if } x = \{b\} \\ 0 & \text{otherwise.} \end{cases}$$

We get $\mathtt{m}_{1\oplus 2}(\{a\}) = 0$ *because for any two elements* $x, y \in P(\{a,b,c\})$ *such that* $x \cap y = \{a\}$ *we have* $\mathtt{m}_1(x)\cdot\mathtt{m}_2(y) = 0$*. This result comes from the fact that Dempster–Shafer combination rule simply gets rid of contradiction.*

This conclusion makes sense given the hypothesis, however in a real life context it is counter-intuitive, since the initial assumptions cannot be met. Indeed, it would be more reasonable to conclude that there is 50%-50% that the patient has disease a and/or disease c and that it is very unlikely that it is disease b, because both experts agree on that fact.

Notice that if one decides to assign a very small mass (e.g. 10^{-4}*) instead of* 0 *for* $\mathtt{m}_1(\{c\})$

and $\mathtt{m}_2(\{a\})$*, one gets the following mass functions*

$$\mathtt{m}_1 : P(S) \to [0,1] \qquad\qquad \mathtt{m}_2 : P(S) \to [0,1]$$

$$x \mapsto \begin{cases} 0.89995 & \textit{if } x = \{a\} \\ 0.09995 & \textit{if } x = \{b\} \\ 0.0001 & \textit{if } x = \{c\} \\ 0 & \textit{otherwise.} \end{cases} \qquad x \mapsto \begin{cases} 0.0001 & \textit{if } x = \{a\} \\ 0.09995 & \textit{if } x = \{b\} \\ 0.89995 & \textit{if } x = \{c\} \\ 0 & \textit{otherwise.} \end{cases}$$

and the following aggregated mass function:[4]

$$\mathtt{m}_{1\oplus 2}(x) = \begin{cases} 0.00885 & \textit{if } x = \{a\} \\ 0.9823 & \textit{if } x = \{b\} \\ 0.00885 & \textit{if } x = \{c\} \\ 0 & \textit{otherwise.} \end{cases}$$

That is one still concludes that disease b is way more likely than disease a or c. Indeed, one gets $\mathtt{bel}_{1\oplus 2}(b) = 0.9823$ *and* $\mathtt{bel}_{1\oplus 2}(\{a,c\}) = \mathtt{m}_{1\oplus 2}(\{a\}) + \mathtt{m}_{1\oplus 2}(\{c\}) = 0.0177$.

The example above shows how Dempster–Shafer combination rule can give counterintuitive results when interpreted in a classical framework. In what follows we argue that formalising this kind of situations within the framework of BD logic, that is, on De Morgan algebra, will lead to more intuitive conclusions.

Dempster–Shafer combination rule on De Morgan algebras Recall that BD logic is the logic of De Morgan algebras, therefore, in that framework, saying that a formula φ is true means that "we have information supporting that fact that the statement φ is true". Therefore, a priori, no two formulas are contradictory. Indeed, one can have pieces of information supporting contradictory statements. Therefore, if we consider De Morgan algebras, we get the following adaptation of Dempster–Shafer combination rule.

Definition 3.3.2 (Dempster–Shafer combination rule over a De Morgan algebra)**.** *Let* $\mathcal{L}$ *be a De Morgan algebra (without the constants* $\perp$ *and* $\top$ *in the language). Let* $\mathtt{m}_1$ *and* $\mathtt{m}_2$ *be two*

[4]Note that those are rounded numbers.

general mass functions on $\mathcal{L}$. Dempster–Shafer combination rule computes their aggregation $\mathtt{m}_{1\oplus 2}$ as follows.

$$\begin{aligned} \mathtt{m}_{1\oplus 2} &: \mathcal{L} \to [0,1] \\ x &\mapsto \sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 = x\}. \end{aligned} \tag{3.2}$$

Let $\mathcal{L}$ be a bounded De Morgan algebra (that is, with the constants $\perp$ and $\top$ in the language). Let $\mathtt{m}_1$ and $\mathtt{m}_2$ be two mass functions on $\mathcal{L}$. Dempster–Shafer combination rule computes their aggregation $\mathtt{m}_{1\oplus 2}$ as follows.

$$\begin{aligned} \mathtt{m}_{1\oplus 2} &: \mathcal{L} \to [0,1] \\ x &\mapsto \begin{cases} 0 & \text{if } x = \perp \\ \dfrac{\sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 = x\}}{\sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 \neq \perp\}} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

Notice that in equation (3.2), there is no normalisation term. Indeed, here $\sum_{x\in\mathcal{L}} \mathtt{m}_{1\oplus 2}(x) = \sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1, x_2 \in \mathcal{L}\}$. In addition, we have

$$\begin{aligned} \sum_{x\in\mathcal{L}} \mathtt{m}_{1\oplus 2}(x) &= \sum_{x\in\mathcal{L}} \sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 = x\} \\ &= \sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1, x_2 \in \mathcal{L}\} \\ &= \sum_{x_1\in\mathcal{L}} \sum_{x_2\in\mathcal{L}} \mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \\ &= \sum_{x_1\in\mathcal{L}} \mathtt{m}_1(x_1)\cdot \sum_{x_2\in\mathcal{L}} \mathtt{m}_2(x_2). \end{aligned}$$

Therefore, $\mathtt{m}_{1\oplus 2}$ is a general mass function, because $0 \leq \sum\limits_{x_1\in\mathcal{L}} \mathtt{m}_1(x_1) \leq 1$ and $0 \leq \sum\limits_{x_2\in\mathcal{L}} \mathtt{m}_2(x_2) \leq 1$.

Lemma 3.3.1. *In the case of Dempster–Shafer combination rule for bounded De Morgan algebras, if we consider the free De Morgan algebra generated by a finite set of variables* $\mathtt{Prop}$ *and constants $\{\perp, \top\}$, then we have that for every $x \in \mathcal{L}$,*

$$\mathtt{m}_{1\oplus 2}(x) = \sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 = x\}. \tag{3.4}$$

Proof. In equation (3.3), notice that $x_1 \wedge x_2 = \bot$ iff either $x_1 = \bot$ or $x_2 = \bot$. This implies that

$$\begin{aligned}
&\sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 \neq \bot\} \\
&= \sum_{x_1\in\mathcal{L}\smallsetminus\bot}\sum_{x_2\in\mathcal{L}\smallsetminus\bot} \mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \\
&= \sum_{x_1\in\mathcal{L}}\sum_{x_2\in\mathcal{L}} \mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) && \text{(because } \mathtt{m}_1(\bot) = \mathtt{m}_2(\bot) = 0) \\
&= \sum_{x_1\in\mathcal{L}} \mathtt{m}_1(x_1)\cdot\sum_{x_2\in\mathcal{L}} \mathtt{m}_2(x_2) \\
&= \sum_{x_1\in\mathcal{L}} \mathtt{m}_1(x_1) && \text{(because } \textstyle\sum_{x_2\in\mathcal{L}} \mathtt{m}_2(x_2) = 1) \\
&= 1. && \text{(because } \textstyle\sum_{x_1\in\mathcal{L}} \mathtt{m}_1(x_1) = 1)
\end{aligned}$$

Therefore, if $x \neq \bot$, we have $\mathtt{m}_{1\oplus 2}(x) = \sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 = x\}$. If $x = \bot$, then $\sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2) \mid x_1 \wedge x_2 = x\} = 0$, because either $x_1 = \bot$ or $x_2 = \bot$. $\square$

Remark 3.3.1. *In Section 3.4, we introduce* DS *models (Definition 3.4.1) on which we define belief functions over* BD *logic. Notice that we define* $\mathtt{bel}^+$ *and* $\mathtt{bel}^-$*, that generalise the non-standard probabilities* $\mathtt{p}^+$ *and* $\mathtt{p}^-$*, on the Lindenbaum algebra associated to the model. Therefore, we consider belief functions over free De Morgan algebras.*

Example 3.3.2 (Two disagreeing experts. Reasoning with BD logic)**.** *We consider the previous case study, but we consider the general mass functions over the free De Morgan algebra* $\mathcal{DM}_3$ *generated by* $\{a,b,c\}$*:*

$$\begin{aligned}
\mathtt{m}_1 &: \mathcal{DM}_3 \to [0,1] & \mathtt{m}_2 &: \mathcal{DM}_3 \to [0,1] \\
x &\mapsto \begin{cases} 0.9 & \textit{if } x = a \\ 0.1 & \textit{if } x = b \\ 0 & \textit{otherwise.} \end{cases} & x &\mapsto \begin{cases} 0.9 & \textit{if } x = c \\ 0.1 & \textit{if } x = b \\ 0 & \textit{otherwise.} \end{cases}
\end{aligned}$$

We get the following aggregated mass function

$$\mathtt{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \textit{if } x = a \wedge c \\ 0.09 & \textit{if } x = a \wedge b \textit{ or } x = b \wedge c \\ 0.01 & \textit{if } x = b \\ 0 & \textit{otherwise.} \end{cases}$$

Here, one still has $\mathtt{m}_{1\oplus 2}(a) = \mathtt{m}_{1\oplus 2}(c) = 0$*, but* $\mathtt{m}_{1\oplus 2}(b) = 0.01$ *is now small. In addition*

$$\mathtt{bel}_{1\oplus 2}(a) = \mathtt{m}_{1\oplus 2}(a) + \mathtt{m}_{1\oplus 2}(a \wedge b) + \mathtt{m}_{1\oplus 2}(a \wedge c) = 0.9$$

is 4.7 *times larger than*

$$\mathtt{bel}_{1\oplus 2}(b) = \mathtt{m}_{1\oplus 2}(b) + \mathtt{m}_{1\oplus 2}(a \wedge b) + \mathtt{m}_{1\oplus 2}(b \wedge c) = 0.19.$$

Here, the mass function $\mathtt{m}_{1\oplus 2}$ *tells us that the evidence strongly supports the fact that the patient has disease a and c, and that the evidence is less conclusive concerning disease b.*

One could object that in the classical case, it is assumed that it is impossible for the patient to have two diseases, in which case, one might want to formalise the example with the following mass functions:

$$\begin{aligned} \mathtt{m}_1 : \mathcal{DM}_3 &\to [0,1] \\ x &\mapsto \begin{cases} 0.9 & \textit{if } x = a \wedge \neg b \wedge \neg c \\ 0.1 & \textit{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \textit{otherwise.} \end{cases} \end{aligned} \qquad \begin{aligned} \mathtt{m}_2 : \mathcal{DM}_3 &\to [0,1] \\ x &\mapsto \begin{cases} 0.9 & \textit{if } x = \neg a \wedge \neg b \wedge c \\ 0.1 & \textit{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

This gives us the following aggregated mass function:

$$\mathtt{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \textit{if } x = a \wedge \neg a \wedge \neg b \wedge c \wedge \neg c \\ 0.09 & \textit{if } x = a \wedge \neg a \wedge b \wedge \neg b \wedge \neg c \textit{ or } x = \neg a \wedge b \wedge \neg b \wedge c \wedge \neg c \\ 0.01 & \textit{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \textit{otherwise.} \end{cases}$$

Here, the mass function highlights that (1) experts agree the patient does not have disease b and (2) the agent has contradictory information which might lead to the conclusion that *further investigation is necessary. Indeed, if one asks two equally qualified experts about their opinions and if they contradict each other, it is only natural to consult more experts. In the same time, we still have* $\mathtt{bel}_{1\oplus 2}(a) = \mathtt{bel}_{1\oplus 2}(c) = 0.9$ *and* $\mathtt{bel}_{1\oplus 2}(b) = 0.19$.

This framework also has the advantages to allow us to formalise a situation in which both experts did not consider the same set of eventualities. Assume that expert 1 simply did not consider disease c as an option (because they forgot, because they are not aware of it, because they could not test for it...) and expert 2 did not consider disease a as an option. Then the initial mass functions become:

$$\begin{aligned} \mathtt{m}_1 : \mathcal{DM}_3 &\to [0,1] \\ x &\mapsto \begin{cases} 0.9 & \text{if } x = a \wedge \neg b \\ 0.1 & \text{if } x = \neg a \wedge b \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \qquad \begin{aligned} \mathtt{m}_2 : \mathcal{DM}_3 &\to [0,1] \\ x &\mapsto \begin{cases} 0.9 & \text{if } x = \neg b \wedge c \\ 0.1 & \text{if } x = b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

and we get the aggregated mass function

$$\mathtt{m}_{1\oplus 2}(x) = \begin{cases} 0.81 & \text{if } x = a \wedge \neg b \wedge c \\ 0.09 & \text{if } x = a \wedge b \wedge \neg b \wedge \neg c \text{ or } x = \neg a \wedge b \wedge \neg b \wedge c \\ 0.01 & \text{if } x = \neg a \wedge b \wedge \neg c \\ 0 & \text{otherwise.} \end{cases}$$

Here, the conclusion is that the patient is likely to have disease a and c.

In the following, we propose to reason with belief functions within the framework of Belnap Dunn logic. Indeed, this counterintuitive result in the classical framework comes from the fact that classical logic cannot talk about contradictory information. Here, we introduce a slightly generalized notion of belief function that does not require belief of false and true to be 0 and 1. Indeed, within the framework of Belnap Dunn logic, we usually do not have the constant symbols to talk about the false $\bot$ and true $\top$.

Remark 3.3.2. *Notice that many alternatives to Dempster–Shafer combination rule have been proposed. For instance, Dubois and Prade [DP88] have proposed a "hybrid" rule intermediate between the conjunctive and disjunctive sums, in which the product* $\mathtt{m}_1(B) \cdot \mathtt{m}_2(C)$

is assigned to $B\cap C$ whenever $B\cap C\neq\varnothing$, and to $B\cup C$ otherwise. This rule is not associative, but it usually provides a good summary of partially conflicting items of evidence. This rule deals with contradictory evidence by stating that at least one of the two options supported by the two pieces of evidence must be true, while Dempster–Shafer combination rule disregards the two pieces of evidence. In our generalisation of Dempster–Shafer combination rule (Definition 3.3.2), we end up keeping track of the contradictions and where they come from. However, notice that if we work on an arbitrary bounded De Morgan algebra, we could still have cases where $x\wedge y=\bot$ even though neither $x=\bot$ nor $y=\bot$. This would happen if, even though the agent is tolerant to contradiction, they consider that it is impossible to get information that x and y are both true. Therefore, in this situation it could make sense to use the combination rule proposed by Dubois and Prade, but on a De Morgan algebra rather than on a powerset algebra:

$$\begin{aligned}\mathtt{m}_{1\oplus 2}(x) = &\sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2)\mid x_1\wedge x_2=x\}\\ &+\sum\{\mathtt{m}_1(x_1)\cdot\mathtt{m}_2(x_2)\mid x_1\wedge x_2=\bot \textit{ and } x_1\vee x_2=x\}.\end{aligned}\tag{3.5}$$

3.4 Two-dimensional reading of belief and plausibility

In this section, we introduce DS models on which we define belief and plausibility of formulas. Then we discuss different interpretations of the notions of belief and plausibility within the two-dimensional treatment of evidence of BD logic.

Definition 3.4.1 (DS models and their associated belief functions)**.** *Let $\mathcal{L}_{\mathsf{BD}}$ be the Lindenbaum algebra for Belnap–Dunn logic over the set of propositional letters* $\mathtt{Prop}$*. A* DS model *is a tuple $\mathfrak{M}=\langle S,P(S),\mathtt{bel},v^+,v^-\rangle$ such that $\langle S,v^+,v^-\rangle$ is a* BD *model and* $\mathtt{bel}$ *is a belief function on $P(S)$. We denote $\mathtt{bel}^+_{\mathfrak{M}}:\mathcal{L}_{\mathsf{BD}}\to[0,1]$ and $\mathtt{bel}^-_{\mathfrak{M}}:\mathcal{L}^{op}_{\mathsf{BD}}\to[0,1]$ the maps such that, for every $\varphi\in\mathcal{L}_{\mathsf{BD}}$,*

$$\mathtt{bel}^+_{\mathfrak{M}}(\varphi)=\mathtt{bel}(|\varphi|^+)\qquad\textit{and}\qquad\mathtt{bel}^-_{\mathfrak{M}}(\varphi)=\mathtt{bel}(|\varphi|^-)=\mathtt{bel}(|\neg\varphi|^+)=\mathtt{bel}^+_{\mathfrak{M}}(\neg\varphi).\tag{3.6}$$

We drop the subscript whenever there is no ambiguity on the model $\mathfrak{M}$ we are considering.

In the classical case (i.e., on a Boolean algebra $\mathcal{B}$), one can define plausibility and belief function in terms of one another or via the mass function associated to the belief in several ways. This is why, there is no need to define *both* plausibility and belief on *classical* Dempster–Shafer structures. Indeed, the plausibility of $a \in \mathcal{B}$ can be construed as the lack of belief in its negation:

$$\mathtt{pl}(a) = 1 - \mathtt{bel}({\sim}a) \tag{3.7}$$

or, equivalently, as the sum of the mass of every statement compatible with a:

$$\mathtt{pl}(a) = \sum_{a \wedge b \neq \bot} \mathtt{m}(b). \tag{3.8}$$

Equations (3.7) and (3.8) are, however, not equivalent on unbounded and free De Morgan algebras. Consider a free De Morgan algebra $\mathcal{A}$ and a (general) mass function $\mathtt{m}$ on $\mathcal{A}$. If $\mathcal{A}$ is unbounded, (3.8) cannot be defined. If $\mathcal{A}$ is bounded, it gives $\mathtt{pl}(a) = 1$ for every $a \in \mathcal{A}$. Indeed, if $\mathcal{A}$ is free, then $a \wedge b = \bot$ iff $a = \bot$ or $b = \bot$. But $\mathtt{m}(\bot) = 0$, whence $\mathtt{pl}(a) = 1$ for any $a \neq \bot$. On the other hand, (3.7) does not necessarily equal to 1 on every $a \neq \bot$ should one substitute Boolean negation $\sim$ for the De Morgan $\neg$.

Definition 3.4.2 ($\mathsf{DS}_{\mathtt{pl}}$ models and their associated plausibility functions)**.** *Let $\mathcal{L}_{\mathsf{BD}}$ be the Lindenbaum algebra for Belnap–Dunn logic over the set of propositional letters* $\mathtt{Prop}$. *A* $\mathsf{DS}_{\mathtt{pl}}$ model *is a tuple $\mathfrak{M} = \langle S, P(S), \mathtt{bel}, \mathtt{pl}, v^+, v^- \rangle$ such that $\langle S, P(S), \mathtt{bel}, v^+, v^- \rangle$ is a* DS *model,* $\mathtt{pl}$ *is a plausibility function on $P(S)$. We denote $\mathtt{pl}^+_{\mathfrak{M}} : \mathcal{L}_{\mathsf{BD}} \to [0,1]$ and $\mathtt{pl}^-_{\mathfrak{M}} : \mathcal{L}^{op}_{\mathsf{BD}} \to [0,1]$ the maps such that, for every $\varphi \in \mathcal{L}_{\mathsf{BD}}$,*

$$\mathtt{pl}^+_{\mathfrak{M}}(\varphi) = \mathtt{pl}(|\varphi|^+) \qquad \textit{and} \qquad \mathtt{pl}^-_{\mathfrak{M}}(\varphi) = \mathtt{pl}(|\varphi|^-) = \mathtt{pl}(|\neg\varphi|^+). \tag{3.9}$$

We drop the subscript whenever there is no ambiguity on the model $\mathfrak{M}$ we are considering.

Note that like in the case of non-standard probabilities (see Lemma 1.4.1), it is equivalent to define belief and plausibility on the Lindenbaum algebra or on the set of formulas.

Lemma 3.4.1. *Let $\mathfrak{M} = \langle S, P(S), \mathtt{bel}, \mathtt{pl}, v^+, v^- \rangle$ be a* $\mathsf{DS}_{\mathtt{pl}}$ *model.* $\mathtt{bel}^+$ *(resp.,* $\mathtt{pl}^+$*) is a general belief (resp., plausibility) function on the Lindenbaum algebra.* $\mathtt{bel}^-$ *(resp.,* $\mathtt{pl}^-$*) is a* *general belief (resp., plausibility) function on the dual of the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}^{op}$.*

Proof. $\mathtt{bel}$ and $|\cdot|^+$ are monotone maps, therefore $\mathtt{bel}^+$ is monotone. In addition, for every $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_{\mathsf{BD}}$, we have

$$\begin{aligned}
\mathtt{bel}^+\left(\bigvee_{1\leq i\leq k}\varphi_i\right) &= \mathtt{bel}\left(\left|\bigvee_{1\leq i\leq k}\varphi_i\right|^+\right) = \mathtt{bel}\left(\bigcup_{1\leq i\leq k}|\varphi_i|^+\right) \\
&\geq \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{bel}\left(\bigcap_{j\in J}|\varphi_j|^+\right) \\
&= \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{bel}\left(\left|\bigwedge_{j\in J}\varphi_j\right|^+\right) \\
&= \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot\mathtt{bel}^+\left(\bigwedge_{j\in J}\varphi_j\right).
\end{aligned}$$

The proof for $\mathtt{pl}^+$ is similar. Now let $\varphi, \psi \in \mathcal{L}_{\mathsf{BD}}^{op}$ such that $\varphi \leq \psi$, that is, such that $\psi \vdash_{\mathsf{BD}} \varphi$. $|\cdot|^-$ is order-reversing w.r.t. $\vdash_{\mathsf{BD}}$, therefore, $|\varphi|^- \leq |\psi|^-$. $\mathtt{bel}$ is a monotone map, therefore $\mathtt{bel}^-(\varphi) = \mathtt{bel}(|\varphi|^-) \leq \mathtt{bel}(|\psi|^-) = \mathtt{bel}^-(\psi)$. Therefore $\mathtt{bel}^-$ is monotone. In addition, for every $\varphi_1, \ldots, \varphi_n \in \mathscr{L}_{\mathsf{BD}}$, we have

$$\begin{aligned}
\mathtt{bel}^-\left(\bigvee^{op}_{1\le i\le k} \varphi_i\right) &= \mathtt{bel}\left(\left|\bigvee^{op}_{1\le i\le k} \varphi_i\right|^-\right) \qquad\qquad (\textstyle\bigvee^{op} \text{ denotes the join of } \mathcal{L}^{op}_{\mathsf{BD}})\\
&= \mathtt{bel}\left(\left|\bigwedge_{1\le i\le k} \varphi_i\right|^-\right) = \mathtt{bel}\left(\bigcup_{1\le i\le k} |\varphi_i|^-\right)\\
&\ge \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{bel}\left(\bigcap_{j\in J} |\varphi_j|^-\right)\\
&= \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{bel}\left(\left|\bigvee_{j\in J} \varphi_j\right|^-\right)\\
&= \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{bel}\left(\left|\bigwedge^{op}_{j\in J} \varphi_j\right|^-\right)\\
&= \sum_{\substack{J\subseteq\{1,\ldots,k\}\\ J\neq\varnothing}} (-1)^{|J|+1}\cdot \mathtt{pl}^-\left(\bigwedge^{op}_{j\in J} \varphi_j\right).
\end{aligned}$$

Therefore $\mathtt{bel}^-$ is a general plausibility function on the dual of the Lindenbaum algebra. The proof for $\mathtt{pl}^-$ is similar. □

The previous lemma shows that each DS model generates a function on the Lindenbaum, and by extension on the set of formulas of BD logic, that satisfies the axioms of (general) belief functions from Definition 1.3.3. The following theorem shows that the converse holds as well: for every (general) belief function on BD formulas, and by extension on the Lindenbaum algebra, we can define a canonical DS model equipped with a belief function such that both functions correspond.

Theorem 3.4.1 (Completeness of belief axioms)**.** *Let* $\mathtt{bel}$ *be a function on* BD *formulas satisfying the axioms of (general) belief function (see Definition 1.3.3). Then there is a canonical model* $\mathfrak{M}_c$ *and a belief function* $\mathtt{bel}'$ *on the powerset of states of* $\mathfrak{M}_c$ *such that* $\mathtt{bel}(\varphi) = \mathtt{bel}'(|\varphi|^+)$.

Proof. The proof goes along the same lines as the one of Theorem 1.4.1. We start with the canonical model $\mathfrak{M}_c = \langle P(\mathsf{Lit}), v_c^+, v_c^- \rangle$ from Definition 3.2.1. The general belief function `bel` on BD formulas can be equivalently represented as a general belief function `bel` on the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}$. Using this general belief function `bel` on the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}$, we define the belief function $\mathtt{bel}^*$ on the uppersets of the poset $\langle P(\mathsf{Lit}), \subseteq \rangle$ (we assign $\mathtt{bel}^*(\varnothing) = 0$ and $\mathtt{bel}^*(P(\mathsf{Lit})) = 1$). Then, we finish the proof by applying Lemma 1.3.7. Notice that unlike in the case of probabilities this lemma does not guarantee uniqueness of the extension of $\mathtt{bel}^*$. $\square$

The proof of the completeness of the plausibility axioms (Theorem 3.4.2) relies on the following remark that allows us to define a De Morgan negation on the powerset of the domain of the canonical model that coincides with the BD negation on the extensions of formulas.

Remark 3.4.1 (De Morgan negation on $P(P(\mathsf{Lit}))$)**.** *Let $w \subseteq \mathsf{Lit}$. We build the set $\mathsf{Lit}^{\mathbf{4}}(w)$ as follows*

$$\forall p \in \mathtt{Prop}: \begin{cases} \mathbf{T}(p) \in \mathsf{Lit}^{\mathbf{4}}(w) & \textit{iff } p \in w, \neg p \notin w \\ \mathbf{B}(p) \in \mathsf{Lit}^{\mathbf{4}}(w) & \textit{iff } p, \neg p \in w \\ \mathbf{N}(p) \in \mathsf{Lit}^{\mathbf{4}}(w) & \textit{iff } p, \neg p \notin w \\ \mathbf{F}(p) \in \mathsf{Lit}^{\mathbf{4}}(w) & \textit{iff } p \notin w, \neg p \in w \end{cases} \tag{3.10}$$

E.g., if $\mathtt{Prop} = \{p, q, r\}$ and $w = \{\neg p, q, \neg q\}$, then $\mathsf{Lit}^{\mathbf{4}}(w) = \{\mathbf{F}(p), \mathbf{B}(q), \mathbf{N}(r)\}$. We call $\mathbf{X}p$'s '$\mathbf{4}$-literals'.

It is clear that, for any $A \in P(P(\mathsf{Lit}))$, there is a unique (up to permutations) disjunctive normal form $\mathsf{Fm}(X)$ whose conjunctive clauses contain $\mathbf{4}$-literals and that every BD formula φ is represented by exactly one $A \in P(P(\mathsf{Lit}))$ which we denote $\mathsf{S}(\varphi)$.

We now need to define a proper De Morgan negation on $\langle P(P(\mathsf{Lit})), \subseteq \rangle$ that extends the BD negation on $\mathscr{L}_{\mathsf{BD}}$ formulas. We take $A \subseteq P(\mathsf{Lit})$ and then $\mathsf{Fm}(A)$. Now we transform $\neg\mathsf{Fm}(A)$ into its disjunctive normal form using the following additional rules

$$\neg\mathbf{X}(p) \rightsquigarrow [\neg_{\mathbf{4}}\mathbf{X}](p) \qquad \mathbf{X}(p) \wedge \mathbf{Y}(p) \rightsquigarrow [\mathbf{X} \wedge_{\mathbf{4}} \mathbf{Y}](p) \qquad \mathbf{X}(p) \vee \mathbf{Y}(p) \rightsquigarrow [\mathbf{X} \vee_{\mathbf{4}} \mathbf{Y}](p)$$

with $\neg_4$, $\vee_4$, and $\wedge_4$ following the truth-table definitions of negation, disjunction, and conjunction in BD. *Notice that if $A = \mathsf{S}(\varphi)$ for some $\varphi \in \mathscr{L}_{\mathsf{BD}}$, then $\neg A = \mathsf{S}(\neg\varphi)$.*

Theorem 3.4.2 (Completeness of plausibility axioms)**.** *Let* $\mathtt{pl}$ *be a function on* BD *formulas satisfying the axioms of (general) plausibility function (see Definition 1.3.5). Then, there is a canonical model $\mathfrak{M}_c$ and a plausibility function* $\mathtt{pl}'$ *on the powerset of states of $\mathfrak{M}_c$ such that* $\mathtt{pl}(\varphi) = \mathtt{pl}'(|\varphi|^+)$.

Proof. We start with the canonical model $\mathfrak{M}_c = \langle P(\mathsf{Lit}), v_c^+, v_c^- \rangle$ from Definition 3.2.1. Let $\mathcal{P}^{\uparrow}(\mathsf{Lit})$ denote the distributive lattice generated by the upsets of $\langle P(\mathsf{Lit}), \subseteq \rangle$. The general plausibility function $\mathtt{pl}$ on BD formulas can be equivalently represented as a general plausibility function $\mathtt{pl}$ on the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}$. Using this general plausibility function $\mathtt{pl}$ on the Lindenbaum algebra $\mathcal{L}_{\mathsf{BD}}$, we define the plausibility function $\mathtt{pl}^*$ on $\mathcal{P}^{\uparrow}(\mathsf{Lit})$ (we assign $\mathtt{pl}^*(\varnothing) = 0$ and $\mathtt{pl}^*(P(\mathsf{Lit})) = 1$). Since the lattice $\mathcal{P}^{\uparrow}(\mathsf{Lit})$ is isomorphic to the lattice reduct of $\mathcal{L}^*_{\mathsf{BD}}$ (the Lindenbaum algebra for BD^*), we can define a De Morgan negation $\neg_{\mathcal{P}}$ on it that coincides with the negation of $\mathcal{L}^*_{\mathsf{BD}}$. Therefore, $\langle \mathcal{P}^{\uparrow}(\mathsf{Lit}), \cup, \cap, \neg_{\mathcal{P}}, \varnothing, P(\mathsf{Lit}) \rangle$ is a finite bounded De Morgan algebra.

Now, consider the function $\mathtt{bel}^* : \langle \mathcal{P}^{\uparrow}(\mathsf{Lit}), \subseteq \rangle \to [0,1]$ such that $\mathtt{bel}^*(S) = 1 - \mathtt{pl}^*(\neg_{\mathcal{P}} S)$ for every $S \in \mathcal{P}^{\uparrow}(\mathsf{Lit})$. From Lemma 1.3.5, we know that $\mathtt{bel}^*$ is a belief function on $\langle \mathcal{P}^{\uparrow}(\mathsf{Lit}), \subseteq \rangle$. Let $\mathtt{m}^* : \langle \mathcal{P}^{\uparrow}(\mathsf{Lit}), \subseteq \rangle \to [0,1]$ be the mass function of $\mathtt{bel}^*$. We can extend it to $PP(\mathsf{Lit})$ as follows:

$$\mathtt{m}' : PP(\mathsf{Lit}) \to [0,1]$$
$$S \mapsto \begin{cases} \mathtt{m}^*(S) & \text{if } S \in \mathcal{P}^{\uparrow}(\mathsf{Lit}), \\ 0 & \text{otherwise.} \end{cases}$$

The function $\mathtt{m}'$ is clearly a mass function, therefore it defines a belief function $\mathtt{bel}' : PP(\mathsf{Lit}) \to [0,1]$ on the distributive lattice $\langle PP(\mathsf{Lit}), \cup, \cap, \varnothing, P(\mathsf{Lit}) \rangle$. Using Remark 3.4.1, we can extend the De Morgan negation $\neg_{\mathcal{P}}$ to the distributive lattice $\langle PP(\mathsf{Lit}), \cup, \cap, \varnothing, P(\mathsf{Lit}) \rangle$. Therefore, $\mathtt{bel}'$ defines a belief function on the finite De Morgan algebra $\langle PP(\mathsf{Lit}), \cup, \cap, \neg_{\mathcal{P}}, \varnothing, P(\mathsf{Lit}) \rangle$. From Lemma 1.3.6, we know that $\mathtt{pl}' : PP(\mathsf{Lit}) \to [0,1]$ such that $\mathtt{pl}'(S) = 1 - \mathtt{bel}'(\neg_{\mathcal{P}} S)$ is a plausibility function on $(PP(\mathsf{Lit}), \cup, \cap, \neg_{\mathcal{P}}, \varnothing, P(\mathsf{Lit}))$. Therefore, $\mathtt{pl}'$ is also a plausibility function on its lattice reduct $(PP(\mathsf{Lit}), \cup, \cap, \varnothing, P(\mathsf{Lit}))$ and on the underlying Boolean algebra $(PP(\mathsf{Lit}), \cup, \cap, (\cdot)^c, \varnothing, P(\mathsf{Lit}))$. Notice that, for every $\varphi \in \mathscr{L}_{\mathsf{BD}}$, we have $\mathtt{pl}'(|\varphi|^+) = 1 - \mathtt{bel}'(\neg_{\mathcal{P}}|\varphi|^+) = 1 - \mathtt{bel}^*(\neg_{\mathcal{P}}|\varphi|^+) = \mathtt{pl}^*(|\varphi|^+) = \mathtt{pl}(\varphi)$. $\square$

We have introduced BD models equipped with belief and plausibility functions. Here, we propose different ways to combine belief and plausibility with a two-dimensional interpretation in order to introduce modalities in the logical formalism.

Belnapian belief

We consider the following two-dimensional reading of belief. We look at belief as a generalisation of non-standard probabilities, where the import-export axiom (see axiom (iii) of Definition 1.4.2) is weakened to the property of being weakly totally monotone (see Definition 1.3.3). If we consider a probabilistic BD model $\mathfrak{M} = \langle W, \mu, v^+, v^- \rangle$, then the two-dimensional value of the probability of a formula φ is $(\mu(|\varphi|^+), \mu(|\varphi|^-))$ and it is interpreted as follows. Positive probability $\mu(|\varphi|^+)$ is the degree to which evidence supports truth of φ, while negative probability $\mu(|\varphi|^-)$ is the degree to which evidence supports its falsity (which is the same as positive probability of $\neg\varphi$).

Following these lines we define a Belnapian belief $(\mathtt{bel}(|\varphi|^+), \mathtt{bel}(|\varphi|^-))$ based on a DS model $\mathfrak{M}$. Analogously to the case of non-standard probabilities the value $\mathtt{bel}(|\varphi|^+)$ represents the degree to which the evidence supports φ and $\mathtt{bel}(|\varphi|^-)$ represents the degree to which the evidence supports its negation. A natural way to introduce plausibility of a formula φ is to use the classical definition via the belief of the negation of φ. This definition is correct, because we know from Lemma 1.3.6 that since $\mathtt{bel}^+$ is a general belief function on $\mathcal{L}_{\mathsf{BD}}$, then the map $\mathtt{pl}^+$ defined as $\mathtt{pl}^+(\varphi) = 1 - \mathtt{bel}^+(\neg\varphi)$ is a general plausibility function on $\mathcal{L}_{\mathsf{BD}}$. Similarly, $\mathtt{pl}^-(\varphi) = 1 - \mathtt{bel}^-(\neg\varphi)$ is a general plausibility function on $\mathcal{L}_{\mathsf{BD}}^{op}$. Observe that in the case of strong contradictory belief in some proposition it can happen that plausibility is strictly smaller than belief, contrary to the intuition understanding them as upper and lower bound.

Formally, we can work with belnapian plausibility and take the pair $(\mathtt{pl}^+(\varphi), \mathtt{pl}^-(\varphi))$ as the primary notion while belief would be a derived one, but this choice is less appealing from the point of view of an interpretation. Indeed, in our framework, one can separate pieces of evidence within three categories: classical (where the evidence for and evidence against add up exactly to 1), incomplete (the evidence for and evidence against add up to a number smaller than 1), or contradictory (the evidence for and evidence against add up to a number greater than 1). Each of these gives us different signals. In particular, classical information might be intuitively interpreted as indicative of us being on the right track in the investigation. I.e., if the information on p is classical, then investigation into p can be deemed *satisfactory*. In this vein, incomplete information can be interpreted as us needing to *investigate p further*, while contradictory information on p shows us that our previous investigation was faulty, whence we need to *re-investigate once again*.

Combining belief and plausibility

Independence of positive and negative support which is in the key idea of the Belnap-Dunn approach gives us more freedom in combining belief and plausibility than in the classical approach. While in the previous case both positive and negative supports were represented by the same uncertainty measure, now we discuss the possibility of combining them. The idea behind the classical belief–plausibility relation is that we can see plausibility of a proposition as a lack of support for its negation. This motivates our second choice for the representation of negative support of a proposition: it is not a (straightforward) support of its negation as in the previous case, but rather a lack of support of the proposition itself.

Formally, we consider a $\mathsf{DS}_{\mathtt{pl}}$ model containing both belief and plausibility (in general they might be computed from different mass function, so they are not mutually definable) and define the positive and negative support pair as $(\mathtt{bel}^{+}(\varphi), \mathtt{pl}^{-}(\varphi)) = (\mathtt{bel}^{+}(\varphi), \mathtt{pl}^{+}(\neg\varphi))$.

Belief and plausibility as lower and upper bounds

In this section, we focus on the interpretation of belief and plausibility as a lower and an upper approximation of the probability of a formula. To do so, we have to consider $\mathsf{DS}_{\mathtt{pl}}$ models $\mathfrak{M} = \langle S, P(S), \mathtt{bel}, \mathtt{pl}, v^{+}, v^{-} \rangle$ with both belief and plausibility introduced independently, and study the implications of the following property:

$$\texttt{bel}(Y) \leq \texttt{pl}(Y), \text{ for every } Y \subseteq P(S). \tag{3.11}$$

Within the framework of BD logic, this immediately implies that one cannot define $\texttt{pl}$ via $\texttt{bel}$ without imposing strong constraints on the valuation of the BD model. Therefore, here we are studying the meaning of having a general belief function and a general plausibility function generated by two different mass functions. First, notice that equation (3.11) is equivalent to having for every v^+:

$$\sum_{X \subseteq |p|^+} \mathtt{m}_{\texttt{bel}}(X) \leq 1 - \sum_{X \subseteq |\neg p|^+} \mathtt{m}_{\texttt{pl}}(X). \tag{3.12}$$

Indeed, from Lemma 1.3.4 and Lemma 1.3.5, we have

$$\texttt{bel}(|p|^+) = \sum_{X \subseteq |p|^+} \mathtt{m}_{\texttt{bel}}(X) \leq \texttt{pl}(|p|^+) = 1 - \texttt{bel}_{\texttt{pl}}(|\neg p|^+) = 1 - \sum_{X \subseteq |\neg p|^+} \mathtt{m}_{\texttt{pl}}(X).$$

In addition, since $\texttt{bel}$ and $\texttt{pl}$ are respectively belief and plausibility functions, then we get the constraint

$$\texttt{bel}(|p|^+) = \sum_{X \subseteq |p|^+} \mathtt{m}_{\texttt{bel}}(X) \leq \sum_{X \not\subseteq |p|^-} \mathtt{m}_{\texttt{pl}}(X) = \texttt{pl}(|p|^+), \tag{3.13}$$

because

$$\begin{aligned}
\texttt{bel}(|p|^+) = \sum_{X \subseteq |p|^+} \mathtt{m}_{\texttt{bel}}(X) \leq \texttt{pl}(|p|^+) &= 1 - \sum_{X \subseteq |\neg p|^+} \mathtt{m}_{\texttt{pl}}(X) \\
&= \sum_{X \in P(S)} \mathtt{m}_{\texttt{pl}}(X) - \sum_{X \subseteq |\neg p|^+} \mathtt{m}_{\texttt{pl}}(X) \\
&= \sum_{X \not\subseteq |\neg p|^+} \mathtt{m}_{\texttt{pl}}(X) = \sum_{X \not\subseteq |p|^-} \mathtt{m}_{\texttt{pl}}(X).
\end{aligned}$$

We consider the following two-dimensional interpretation for belief and plausibility of φ respectively: $B\varphi = (\texttt{bel}^+(\varphi), \texttt{bel}^-(\varphi))$ and $Pl\varphi = (\texttt{pl}^+(\varphi), \texttt{pl}^-(\varphi))$.

Interpretation of the mass functions $\mathtt{m}_{\texttt{bel}}$ and $\mathtt{m}_{\texttt{pl}}$ Here, $B(\varphi) = (x, y)$ is interpreted as follows: $x = \texttt{bel}^+(\varphi)$ is how much the agent is persuaded that φ is true based on the evidence, and $y = \texttt{bel}^-(\varphi)$ is how much the agent is persuaded that φ is false based on the evidence.

Persuasion relies on a wide range of evidence: reliable scientific evidence, but also emotional reaction to an argument. Therefore, one could ignore the contradictoriness of some evidence to strengthen one's opinion and one can ignore scientific evidence due to some bias. Hence, when computing $\mathtt{m_{bel}}$, the agent uses a weak standard for saying that a piece of evidence supports some statement and is influenced by its own bias: highly contradictory evidence can be considered to support a statement and reliable evidence can be ignored.

However, the agent uses a different standard of evidence to decide that a statement is not plausible. We have $\mathtt{pl}(|p|^+) = \sum_{X \not\subseteq |\neg p|^+} \mathtt{m_{pl}}(X)$, therefore, p is considered plausible if there is very little strong evidence supporting $\neg p$. We interpret this as follows: the agent considers p plausible, if they are not convinced that $\neg p$ is the case. Conviction is built on 'reliable' evidence. The meaning reliable will depend on the context: in a court, it could be the kind of evidence accepted by the court, in science it could be detailed proofs that have been reviewed by experts... One can for instance believe that a mathematical statement is false because of some personal intuition based on experience, even if there is a verified proof of the statement which makes it not very plausible (but not impossible) that it is false. Indeed, with time, some proofs are found to be false.

We propose the following interpretation of the mass functions. $\mathtt{m_{bel}}$ is computed by asking the question "does the evidence persuade the agent?", while $\mathtt{m_{pl}}$ is computed by asking the question "is the evidence considered convincing by some given authority?" In order to get a better intuition on what is going on, let us have a look at some examples.

$$s_0 : \qquad s_1 : p \qquad s_2 : \neg p \qquad s_3 : p, \neg p$$

Figure 3.1: Canonical model over $\mathtt{Prop} = \{p\}$.

Example 3.4.1 (Strong belief in $p \wedge \neg p$)**.** *Consider the set of variables* $\mathtt{Prop} = \{p\}$*. The canonical model is in Figure 3.1. Assume that* $\mathtt{bel}(|p \wedge \neg p|^+) = 1$*. Therefore,*

$$\sum_{X \subseteq |p \wedge \neg p|^+} \mathtt{m_{bel}}(X) = \sum_{X \subseteq \{s_3\}} \mathtt{m_{bel}}(X) = \mathtt{m_{bel}}(\varnothing) + \mathtt{m_{bel}}(\{s_3\}) = 1.$$

Since $\mathtt{bel}$ *is a belief function on* $P(S)$*, we have* $\mathtt{m_{bel}}(\varnothing) = 0$*, therefore* $\mathtt{m_{bel}}(\{s_3\}) = 1$*. This means that based on the available evidence, the agent is persuaded that* $p \wedge \neg p$ *is the case.*

Notice that

$$\begin{aligned}
\mathtt{pl}(|p\wedge\neg p|^{+}) &= \sum_{X\not\subseteq|p\wedge\neg p|^{-}}\mathtt{m}_{\mathtt{pl}}(X) = \sum_{X\not\subseteq|\neg(p\wedge\neg p)|^{+}}\mathtt{m}_{\mathtt{pl}}(X)\\
&= \sum_{X\not\subseteq|\neg p\vee p|^{+}}\mathtt{m}_{\mathtt{pl}}(X) = \sum_{X\not\subseteq|\neg p|^{+}\cup|p|^{+}}\mathtt{m}_{\mathtt{pl}}(X)\\
&= \sum_{X\not\subseteq|p|^{-}\cup|p|^{+}}\mathtt{m}_{\mathtt{pl}}(X)
\end{aligned}$$

Therefore, the condition that $\mathtt{bel}\leq\mathtt{pl}$ *implies that*

$$1=\sum_{X\not\subseteq|p|^{-}\cup|p|^{+}}\mathtt{m}_{\mathtt{pl}}(X)=\sum_{X\not\subseteq\{s_1,s_2,s_3\}}\mathtt{m}_{\mathtt{pl}}(X)=\mathtt{m}_{\mathtt{pl}}(S). \tag{3.14}$$

Therefore, evidence that is strongly persuasive considering $p\wedge\neg p$ *is inconclusive regarding the plausibility of either* p *or* $\neg p$.

Example 3.4.2 (Weak belief in either p or $\neg p$)**.** *We still consider the previous model. Assume now that* $\mathtt{bel}(|p\vee\neg p|^{+})=0$. *Therefore* $\sum_{X\subseteq\{s_1,s_2,s_3\}}\mathtt{m}_{\mathtt{bel}}(X)=0$ *and* $\mathtt{m}_{\mathtt{bel}}(\{s_0,s_1,s_2,s_3\})=1$. *The condition that* $\mathtt{bel}\leq\mathtt{pl}$ *does not constrain the value of* $\mathtt{m}_{\mathtt{pl}}$ *in any way. Therefore, the available evidence might be convincing from the point of view of a given authority, however it did not persuade the agent.*

3.5 Conclusion

As mentioned before, our long term goal is to develop a modular logical framework to reason about crisp and/or graded, incomplete and/or inconsistent information. In the previous chapter, we lay the theoretical ground for that framework, namely the framework of In this chapter, building on the existing work on non-standard probabilities [KMR21], we generalised the theory of belief functions to the framework of Belnap–Dunn logic. The main challenge of that endeavour is to understand how to interpret belief functions and even more plausibility functions. Indeed, in the classical case, a belief function provides a lower bound and an upper bound of the probability of an event. Here, using a De Morgan negation rather than a boolean negation implies that the interpretation of a formula and of its negation are independent.

Therefore, providing an interpretation for these functions becomes more complex. However, answering these questions is key to developing a theory of (imprecise) probabilities over substructural logics.

In the previous section, we discuss how one can combine the belief and the plausibility assigned to a formula φ to form a couple of values that would represent the positive and the negative information available about φ. This work has been the philosophical base to propose two layered logics to reason with belief functions. In [Bíl+22, Section 4], my coauthors provide calculi that formalize reasoning with *both* non-standard probabilities and belief functions. This goal can be reached in two ways. The first one is by defining a calculus that allows for reasoning with statements concerning probabilities or beliefs directly. This is the way it is done in [FHM90]: the calculi there contain three types of axioms and rules: the ones that govern the arithmetical part, i.e., the reasoning about inequalities; the ones that axiomatise probabilities; and the rules and axioms of the logic wherein the reasoning itself occurs, i.e., classical propositional logic. The proposed calculus has the advantage of being quite intuitive and easy to use, however, its axiomatisation is infinite. To address this issue, one can undertake the second approach and utilise a two-layered modal logic in a similar manner to [BCN20a]. A calculus will then consist of three parts: the rules and axioms of the logic of events or 'inner logic'; the 'outer logic' that formalises reasoning with evidence; and finally, the modalities that transform events into probabilistic evidence. While these two approaches may seem different at the first glance, it is shown in [BCN20a] that they are actually equivalent for the classical probabilities when the outer-layer logic is taken to be Łukasiewicz logic. [Bíl+22] provides both these perspectives on reasoning with non-standard probabilities and belief functions and shows that they are equivalent in the same way that the two formalisations of reasoning with classical probabilities are.

CHAPTER 4

Updating belief functions

4.1 Introduction

In the previous chapters, we studied and explained belief and plausibilities on Belnap—Dunn logic. The next natural step is to look at taking new pieces of information into account by updating the belief functions over Belnap–Dunn logic. In fact, [Hal17] presents different ways to update belief and plausibility functions in the classical framework. In this section, we expend our study and discuss how to adapt and interpret those results within the framework of belief and plausibility functions over Belnap–Dunn logic.

We should clarify that, the model that we use allows us to update belief and plausibility on BD logic without any need to face the difficulties of defining and updating them on the Lindenbaum algebras. Probability, belief and plausibility are already defined over distributive lattices. An option to define updating of these functions is to use the congruence lattices (quotient algebras), that is, for a distributive lattice $\mathcal{L}$ and $a \in \mathcal{L}$ we consider the congruence lattice generated by the equivalence relation $\equiv_a$ as follows and we call this lattice $\mathcal{L}_a$:

$$x \equiv_a y \quad \text{iff} \quad x \wedge a = y \wedge a.$$

The problem is that $\mathcal{L}_a$ does not behave well in dealing with negation. In addition, even simple rules such as existing an extension of a probability function from one σ-algebra to a bigger

σ-algebra which we need in defining belief as a lower envelope and then updating them, to our knowledge, does not work from one sub-De-Morgan algebra to a bigger one. We have mentioned in the further works the interesting questions that we should answer and study later.

We proceed as follows. First in Section 4.2, we present we recall some lemmas about belief and plausibility functions and Dempster–Shafer combination rule, and then we present existing proposals to update belief functions over Boolean algebras. In Section 4.3, we introduce models for belief and plausibility over BD logic. We discuss how to update belief and plausibility when getting a new piece of information.

4.2 Preliminaries

In this section, First we explain shortly updating concept and then we present two ways to update belief functions on Boolean algebras. Notice that in this chapter we use the notations `Bel` and `Pl` when our beleif and plausibility functions are defined over a Boolean algebra and we use `bel` and `pl` when de domain is lattice (or a non-Boolean) structure. First we recall a classic definition.

Definition 4.2.1. *a function* $\mu : P(S) \to [0,1]$ *is called a* (finitely additive) probability measure *if it satisfies the following properties:*

1. $\mu(S) = 1$
2. $\mu(A \cup B) = \mu(A) + \mu(B)$ *for* A, B *disjoint.*

We put some definition that we already mentioned and proved in detail to be more reader friendly and to keep the connection of this chapter.

4.2.1 Belief and plausibility functions

For sake of readability, we recall the definitions of belief functions, plausibility functions and mass functions and some lemmas which their proofs are presented in the previous chapter. They are Definition 1.3.3, 1.3.4, 1.3.5 and 3.3.1, and Lemmas 1.3.4, 1.3.6 and 1.3.5, respectively.

Definition 4.2.2 (Belief function)**.** *Let $\mathcal{L}$ be a a bounded lattice. A function* $\mathtt{bel} : \mathcal{L} \to [0,1]$ *is called a* belief function *if the following conditions hold:*

- $\mathtt{bel}(\bot) = 0$ *and* $\mathtt{bel}(\top) = 1$*,*
- $\mathtt{bel}$ *is* monotone *with respect to $\mathcal{L}$: for every $x, y \in \mathcal{L}$, if $x \leq_{\mathcal{L}} y$, then* $\mathtt{bel}(x) \leq \mathtt{bel}(y)$*,*
- $\mathtt{bel}$ *is* weakly totally monotone*: for every $k \geq 1$ and every $a_1, \ldots, a_k \in \mathcal{L}$, it holds that*

$$\mathtt{bel}\left(\bigvee_{1 \leq i \leq k} a_i\right) \geq \sum_{\substack{J \subseteq \{1,\ldots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{bel}\left(\bigwedge_{j \in J} a_j\right). \tag{4.1}$$

Definition [Mass function] Let $\mathcal{L} \neq \varnothing$ be an arbitrary lattice. A *mass function* on $\mathcal{L}$ is a function $\mathtt{m} : \mathcal{L} \to [0,1]$ such that $\sum_{x \in \mathcal{L}} \mathtt{m}(x) = 1$.

Lemma [Mass function associated to a belief function] Let $\mathcal{L}$ be a finite lattice and $\mathtt{bel} : \mathcal{L} \to [0,1]$ a belief function. Then, there is a mass function $\mathtt{m}_{\mathtt{bel}} : \mathcal{L} \to [0,1]$, called the *mass function associated to* $\mathtt{bel}$, such that, for every $x \in \mathcal{L}$,

$$\mathtt{bel}(x) = \sum_{y \leq x} \mathtt{m}_{\mathtt{bel}}(y). \tag{4.2}$$

Conversely, for any mass function $\mathtt{m}$ on the lattice $\mathcal{L}$, the function $\mathtt{bel}_{\mathtt{m}} : \mathcal{L} \to [0,1]$ defined as

$$\mathtt{bel}_m(x) = \sum_{y \leq x} \mathtt{m}(y) \tag{4.3}$$

is a belief function.

Definition [Plausibility functions] Let $\mathcal{L}$ be a bounded lattice. $\mathtt{pl} : \mathcal{L} \to [0,1]$ is called a *plausibility function* if the following conditions hold:

- $\mathtt{pl}(\bot) = 0$ and $\mathtt{pl}(\top) = 1$,
- $\mathtt{pl}$ is monotone with respect to $\mathcal{L}$,

- for every $k \geq 1$ and every $a_1, \dots, a_k \in \mathcal{L}$, it holds that

$$\mathtt{pl}\left(\bigwedge_{1\leq i\leq k} a_i\right) \leq \sum_{\substack{J \subseteq \{1,\dots,k\} \\ J \neq \varnothing}} (-1)^{|J|+1} \cdot \mathtt{pl}\left(\bigvee_{j\in J} a_j\right). \tag{4.4}$$

Lemma [Plausibility function associated to a belief function] Let $\mathcal{L}$ be a bounded De Morgan algebra and $\mathtt{bel} : \mathcal{L} \to [0,1]$ a belief function. Then, the function $\mathtt{pl}_{\mathtt{bel}} : \mathcal{L} \to [0,1]$ such that $\mathtt{pl}_{\mathtt{bel}}(x) = 1 - \mathtt{bel}(\neg x)$ is a plausibility function, called the *plausibility function associated to* $\mathtt{bel}$.

Lemma [Mass function associated to a plausibility function] Let $\mathcal{L}$ be a bounded De Morgan algebra, and $\mathtt{pl} : \mathcal{L} \to [0,1]$ a plausibility function. Then, the function $\mathtt{bel}_{\mathtt{pl}} : \mathcal{L} \to [0,1]$ such that $\mathtt{bel}_{\mathtt{pl}}(x) = 1 - \mathtt{pl}(\neg x)$ is a belief function, called the *belief function associated to* $\mathtt{pl}$. We denote $\mathtt{m}_{\mathtt{pl}}$ the mass function associated to $\mathtt{bel}_{\mathtt{pl}}$ and we call $\mathtt{m}_{\mathtt{pl}}$ the *mass function associated to* $\mathtt{pl}$. Then,

$$\mathtt{pl}(x) = 1 - \sum_{y\leq\neg x} \mathtt{m}_{\mathtt{pl}}(y). \tag{4.5}$$

Notice that, like in the classical case, a mass function $\mathtt{m}$ gives rise to a belief $\mathtt{bel}_{\mathtt{m}}$ and a plausibility $\mathtt{pl}_{\mathtt{m}}$ function such that $\mathtt{pl}_{\mathtt{m}}(x) = 1 - \mathtt{bel}_{\mathtt{m}}(\neg x)$. However, here, since $\neg$ is not a Boolean negation, we cannot prove anymore that $\mathtt{bel}_{\mathtt{m}}(x) \leq \mathtt{pl}_{\mathtt{m}}(x)$. In addition, notice that contrary to the classical case, one cannot rewrite the expression $1 - \sum_{y\leq\neg x} \mathtt{m}_{\mathtt{pl}}(y)$ as $\sum_{y: y\wedge x>\bot} \mathtt{m}_{\mathtt{pl}}(y)$. Indeed, for instance, $\neg x \leq \neg x$, but $\neg x \wedge x \neq \bot$.

Belief functions and their mass functions are used to reason about evidence. Dempster–Shafer combination rule allows merging the information provided by different sources, each source being described by a mass function.

Definition [Dempster–Shafer combination rule]Let $\mathtt{m}_1$ and $\mathtt{m}_2$ be two mass functions on $P(S)$. Dempster–Shafer combination rule computes their aggregation $\mathtt{m}_1 \oplus \mathtt{m}_2 : P(S) \to [0,1]$

as follows.

$$X \mapsto \begin{cases} 0 & \text{if } X = \varnothing \\ \dfrac{\sum\{\mathtt{m}_1(X_1) \cdot \mathtt{m}_2(X_2) \mid X_1 \cap X_2 = X\}}{\sum\{\mathtt{m}_1(X_1) \cdot \mathtt{m}_2(X_2) \mid X_1 \cap X_2 \neq \varnothing\}} & \text{otherwise.} \end{cases}$$

4.2.2 Classical updating of uncertainty measures

A probability function μ on a Boolean algebra $P(S)$ represents the information available about the subsets of S representing events. Observing $B \in P(S)$ changes the probabilities assigned to the elements of $P(S)$ and leads to a new probability measure μ_B. There are different strategies to define this new measure. Here we focus on Bayesian updating defined as $\mu_B(C) = \frac{\mu(B \cap C)}{\mu(B)}$, for every $C \in P(S)$. This formula comes from assuming the following conditions for the Bayesian updating, as you cans see in [Hal17];

1. $\mu_B(\overline{B}) = 0$, the worlds in $\overline{B}$ are impossible.

2. $\frac{\mu(V_1)}{\mu(V_2)} = \frac{\mu_B(V_1)}{\mu_B(V_2)}$ for $V_1, V_2 \subseteq B$, relative likelihood of worlds in B must remain unchanged.

In this section, we present results from [Hal17]. We recall how Bayesian updating is used to define conditional upper and lower probabilities, and conditional belief and plausibility, and how Dempster-Shafer combination rule can be used to define conditional belief and plausibility. In this section, $\mathtt{Pl}$ denotes the plausibility function associated to $\mathtt{Bel}$.

Conditioning upper and lower probabilities

Let $\mathcal{A}$ be a non-empty set of probability measures defined over $P(S)$. Then its *lower and upper probabilities* (resp. $\mathcal{A}_*$ and $\mathcal{A}^*$) are functions defined on $P(S)$ as follows, for every $X \in P(S)$:

$$\mathcal{A}_*(X) = \inf\{\mu(X) : \mu \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}^*(X) = \sup\{\mu(X) : \mu \in \mathcal{A}\}. \tag{4.6}$$

Halpern in [Hal17] proposes the following way to update a set of probabilities $\mathcal{A}$ using Bayesian update. A priori, observing $B \in P(S)$ leads to the Bayesian updating μ_B of all probabilities $\mu \in \mathcal{A}$ such that $\mu(B) \neq 0$, that is, the probability measures consistent with that observation. Therefore, the update of $\mathcal{A}$ is defined only if there is at least one $\mu' \in \mathcal{A}$ such that

$\mu'(B) > 0$. We denote $\mathcal{A}_B = \{\mu_B : \mu \in \mathcal{A} \text{ and } \mu(B) > 0\}$. We define the *conditional updating of $\mathcal{A}_*$ and $\mathcal{A}^*$ by B* as follows: $(\mathcal{A}^*)_B := (\mathcal{A}_B)^*$ and $(\mathcal{A}_*)_B := (\mathcal{A}_B)_*$. The second option is to define the conditioning only if $\mu(B) > 0$ for every $\mu \in \mathcal{A}$. This condition is a stronger condition so conditioning on less number of sets is possible. We can interpret this definition as follows: assuming that each probability measure is a measure given by a source, then we can condition on B if there is at least one source that consider it possible, while the latter means we can condition on B if all the sources consider it possible, i.e. $\mu(B) > 0$ for every μ.

Conditioning belief and plausibility as lower and upper probabilities

In the classical case, [Hal17] introduces two ways to update belief functions: (1) via the representation of belief functions as lower probabilities (see Theorem 4.2.1), and (2) via their associated mass functions (see Proposition 4.2.2). One of the theorems that we need is the following theorem which characterises `bel` based on upper and lower probabilities.

Theorem 4.2.1. *[Hal17, Theorem 2.6.1] Let* `Bel` *be a belief function defined on $P(S)$ and $\mathscr{M}_{\texttt{Bel}} = \{\mu : \mu(X) \geq \texttt{Bel}(X)$, for all $X \in P(S)\}$. Then* $\texttt{Bel} = (\mathscr{M}_{\texttt{Bel}})_*$ *and* $\texttt{Pl} = (\mathscr{M}_{\texttt{Bel}})^*$.

The set $\mathscr{M}_{\texttt{Bel}}$ can be updated when it contains at least one measure such that $\mu(B) > 0$, that is when $\texttt{Pl}_{\texttt{Bel}}(B) > 0$.

Definition 4.2.3. *Let* $\texttt{Bel} : P(S) \to [0,1]$ *be a belief function and* `Pl` *is associated plausibility function such that* $\texttt{Pl}(X) = 1 - \texttt{Bel}(\overline{X})$ *for every $X \in P(S)$. Then* conditioning of `Bel` and `Pl` on *B is defined as follows:* $\texttt{Bel}_B(X) = ((\mathscr{M}_{\texttt{Bel}})_*)_B(X) = ((\mathscr{M}_{\texttt{Bel}})_B)_*(X)$ *and* $\texttt{Pl}_B(X) = ((\mathscr{M}_{\texttt{Bel}})^*)_B(X) = ((\mathscr{M}_{\texttt{Bel}})_B)^*(X)$.

In section 4.3.3 and as we discussed in the previous chapter, we will work with models containing belief and plausibility functions that are not interdefinable. The following proposition characterises those functions in terms of lower and upper probabilities.

Proposition 4.2.1 (Belief and plausibility as an upper/lower probability)**.** *Let f be a function defined on $P(S)$ and $\mathscr{M}_f = \{\mu : \mu(X) \geq f(X)$, for all $X \in P(S)\}$ and $\mathscr{N}_f = \{\mu : \mu(X) \leq f(X)$, for all $X \in P(S)\}$. Now let* `Bel` *and* `Pl` *be belief and plausibility functions defined* *independently on $P(S)$. Then,*

$$\mathtt{Pl} = (\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})^* = (\mathscr{N}_{\mathtt{Pl}})^* \qquad \textit{and} \qquad \mathtt{Bel} = (\mathscr{M}_{\mathtt{Bel}})_* = (\mathscr{N}_{\mathtt{Pl}_{\mathtt{Bel}}})_*$$

where $\mathtt{Bel}_{\mathtt{Pl}}$ and $\mathtt{Pl}_{\mathtt{Bel}}$ are respectively the belief and plausibility associated to $\mathtt{Pl}$ and $\mathtt{Bel}$ (see Lemmas 1.3.6 and 1.3.5).

Proof. Recall that $\mathtt{Bel}_{\mathtt{Pl}}(X) = 1 - \mathtt{Pl}(\overline{X})$. Based on Theorem 4.2.1, we have $\mathtt{Bel}_{\mathtt{Pl}} = (\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})_*$ and $\mathtt{Pl} = (\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})^*$. Notice that, for every $X \in P(S)$,

$$\begin{aligned}
\mu(X) \le \mathtt{Pl}(X) &\iff \mu(X) \le 1 - \mathtt{Bel}_{\mathtt{Pl}}(\overline{X}) \\
&\iff \mathtt{Bel}_{\mathtt{Pl}}(\overline{X}) \le 1 - \mu(X) \\
&\iff \mathtt{Bel}_{\mathtt{Pl}}(\overline{X}) \le \mu(\overline{X}) \\
&\iff \mathtt{Bel}_{\mathtt{Pl}}(X) \le \mu(X) \\
&\iff \mathtt{Bel}_{\mathtt{Pl}}(X) \le \mu(X)
\end{aligned}$$

Therefore,

$$\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}} = \mathscr{N}_{\mathtt{Pl}} \quad \text{and} \quad (\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})^* = (\mathscr{N}_{\mathtt{Pl}})^*$$

The proof for $\mathtt{Bel}$ is similar. $\square$

Defining conditioning via lower and upper probabilities is not very practical from a computational perspective. The following theorem gives us an explicit formula.

Theorem 4.2.2. *Let $\mathtt{Bel} : P(S) \to [0,1]$ be a belief function. Let $\mathtt{Pl}$ be its associated plausibility function. Suppose that $\mathtt{Pl}(B) > 0$. Then,*

$$\mathtt{Bel}_B(X) = \begin{cases} 1 & \textit{if } \mathtt{Pl}(\overline{X} \cap B) = 0, \\ \dfrac{\mathtt{Bel}(X \cap B)}{\mathtt{Bel}(X \cap B) + \mathtt{Pl}(\overline{X} \cap B)} & \textit{if } \mathtt{Pl}(\overline{X} \cap B) > 0. \end{cases}$$

$$\mathtt{Pl}_B(X) = \frac{\mathtt{Pl}(X \cap B)}{\mathtt{Pl}(X \cap B) + \mathtt{Pl}(\overline{X} \cap B)}$$

Proof. [Hal17, Theorem 3.8.2] proves the formula for $\mathtt{Bel}$ and the fact that

$$\mathtt{Pl}_B(X) = \begin{cases} 0 & \text{if } \mathtt{Pl}(X\cap B) = 0, \\ \dfrac{\mathtt{Pl}(X\cap B)}{\mathtt{Pl}(X\cap B)+\mathtt{Pl}(\overline{X}\cap B)} & \text{if } \mathtt{Pl}(X\cap B) > 0. \end{cases}$$

Notice that $\mathtt{Pl}(X\cap B)+\mathtt{Pl}(\overline{X}\cap B) > 0$ for every $X \in P(S)$. Indeed, since $\mathtt{Pl}$ is a plausibility function, we have $\mathtt{Pl}(A\cup C) \leq \mathtt{Pl}(A)+\mathtt{Pl}(C)-\mathtt{Pl}(A\cap C)$ for every $A, C \in P(S)$. If $A\cap C = \emptyset$, then $\mathtt{Pl}(A\cup C) \leq \mathtt{Pl}(A)+\mathtt{Pl}(C)$. Hence, if $A = \overline{X}\cap B$ and $C = X\cap B$, we have $\mathtt{Pl}(B) \leq \mathtt{Pl}(\overline{X}\cap B)+\mathtt{Pl}(X\cap B)$. Since $\mathtt{Pl}(B) > 0$, we have $\mathtt{Pl}(\overline{X}\cap B)+\mathtt{Pl}(X\cap B) > 0$, for every $X \in P(S)$. Therfore, if $\mathtt{Pl}(X\cap B) = 0$, we have $\frac{\mathtt{Pl}(X\cap B)}{\mathtt{Pl}(X\cap B)+\mathtt{Pl}(\overline{X}\cap B)} = 0$ as requiered. □

Conditioning belief and plausibility via mass functions

In this case, observing B is encoded via the mass function $\mathtt{m}_B$ as $\mathtt{m}_B(B) = 1$ and 0 otherwise. The update $\mathtt{Bel}^B$ of a belief function $\mathtt{Bel}$ by B is computed via Demspter-Shafer combination rule and its associated mass function is $\mathtt{m}_{\mathtt{Bel}} \oplus \mathtt{m}_B$. To distinguish between this method and the above method we use $\mathtt{Bel}^B$ and $\mathtt{Pl}^B$ for the conditional belief and plausibility obtained by the latter approach and we call it DS-conditioning. We have the following explicit formulas for $\mathtt{Bel}^B$ and $\mathtt{Pl}^B$.

Proposition 4.2.2. *[Hal17, Theorem 3.8.5]* $\mathtt{Bel}^B$ *and* $\mathtt{Pl}^B$ *are defined if* $\mathtt{Pl}(B) > 0$. *For every* $X \in P(S)$,

$$\mathtt{Bel}^B(X) = \frac{\mathtt{Bel}(X\cup\overline{B})-\mathtt{Bel}(\overline{B})}{1-\mathtt{Bel}(\overline{B})} \qquad \textit{and} \qquad \mathtt{Pl}^B(X) = \frac{\mathtt{Pl}(X\cap B)}{\mathtt{Pl}(B)}.$$

4.3 Updating belief and plausibility over Belnap-Dunn logic

4.3.1 Models for belief and plausibility over Belnap-Dunn logic

Recalling that a DS *model* is a tuple $\mathfrak{M} = \langle S, P(S), \mathtt{Bel}, v^+, v^- \rangle$ such that $\langle S, v^+, v^- \rangle$ is a BD model and $\mathtt{Bel}$ is a belief function on $P(S)$. From this model as we said we obtain

$\mathtt{bel}^+ : \mathcal{L}_{\mathsf{BD}} \to [0,1]$ and $\mathtt{bel}^- : \mathcal{L}^{op}_{\mathsf{BD}} \to [0,1]$ as follows for every $\varphi \in \mathcal{L}_{\mathsf{BD}}$,

$$\mathtt{bel}^+(\varphi) = \mathtt{Bel}(|\varphi|^+) \qquad \text{and} \qquad \mathtt{bel}^-(\varphi) = \mathtt{Bel}(|\varphi|^-) = \mathtt{Bel}(|\neg\varphi|^+)$$

Recall that as we are defining belief of a formula via its extension, we obtain mutual definability of positive and negative belief: $\mathtt{bel}^-(\varphi) = \mathtt{bel}^+(\neg\varphi)$. This property mirrors how the negation works in BD logic and in non-standard probabilities. It would be possible to define plausibility analogously to the classical case, that is $\mathtt{Pl}(X) = 1 - \mathtt{Bel}(\overline{X})$. The plausibility of φ would then be equal to the sum of the masses of the sets of states that at least partially support φ, i.e.

$$\sum\{m(A) \mid A \cap |\varphi|^+ \neq \emptyset\}. \tag{4.7}$$

This definition does not have an intuitive interpretation, as in the sum (1) we can take into account sets of states that all positively satisfy both φ and $\neg\varphi$ and (2) we do not take into account sets of states that satisfy neither φ nor $\neg\varphi$. However, not having information about φ, in general, is not an argument to say that it is implausible. Therefore, we introduced $\mathsf{DS}_{\mathtt{pl}}$ models where belief and plausibility are not inter-definable as a tuple $\mathfrak{M} = \langle S, P(S), \mathtt{Bel}, \mathtt{Pl}, v^+, v^-\rangle$ such that $\langle S, P(S), \mathtt{Bel}, v^+, v^-\rangle$ is a DS model, $\mathtt{Pl}$ is a plausibility function on $P(S)$. This model gives us $\mathtt{pl}^+ : \mathcal{L}_{\mathsf{BD}} \to [0,1]$ and $\mathtt{pl}^- : \mathcal{L}^{op}_{\mathsf{BD}} \to [0,1]$ the maps such that, for every $\varphi \in \mathcal{L}_{\mathsf{BD}}$,

$$\mathtt{pl}^+(\varphi) = \mathtt{Pl}(|\varphi|^+) \qquad \text{and} \qquad \mathtt{pl}^-(\varphi) = \mathtt{Pl}(|\varphi|^-) = \mathtt{Pl}(|\neg\varphi|^+).$$

In the standard approach both belief and plausibility use in fact the same information represented by the mass function, but deal with it in a different way. While we can see belief as the amount of information which directly supports the statement in question, plausibility represents the amount of information which does not contradict the statement. As Halpern says: "$Plaus_m(U)$ can be thought of as the sum of the probabilities of the evidence that is compatible with the actual world being in U." ([Hal17], p. 38). This idea is captured in the definition of plausibility via mass function: $\mathtt{pl}(A) = \sum\limits_{A\cap B\neq\varnothing} \mathtt{m}(B)$. We can also see belief and plausibility as approximations, as a lower and an upper bound for the 'true' probability: $\mathtt{bel}(A) \leq p(A) \leq \mathtt{pl}(A)$. While in the classical case all these readings coincide, in the case of BD logic they do not, which gives us several possibilities of defining belief/plausibility pairs, which is explained in the previous chapter. Notice that, since A and $\neg A$ are independent elements, if we consider a belief function $\mathtt{bel}$ and its associated plausibility function $\mathtt{pl}_{\mathtt{bel}}$, then we can have $\mathtt{pl}_{\mathtt{bel}}(A) < \mathtt{bel}(A)$. Therefore, asking for $\mathtt{bel}(A) \leq \mathtt{pl}(A)$ usually implies that $\mathtt{bel}$ and $\mathtt{pl}$ are associated to different mass functions. This can be interpreted as follows: an agent has a fixed set of pieces of evidences, however, they do not read the evidence the same way when they ask themselves "does the evidence strongly convince me that φ is the case?" or "is the evidence coherent with the fact that φ might be the case?".

4.3.2 Updating belief

A natural question that arises is what is the behaviour of the positive and negative belief functions induced by the above models, when one learns a new piece of information. Learning something about φ means finding a positive or negative piece of information or even a contradictory piece of information about φ. Here, we directly adapt the conditioning on belief function proposed in [Hal17]. Indeed, the belief function $\mathtt{Bel}$ in a DS models is defined on a powerset algebra. The non-classical behaviour with respect to the negation of $\mathtt{bel}^+$ and $\mathtt{bel}^-$ comes from the non-classical interpretation of formulas. Recall that in BD logic, $|\varphi|^- = |\neg\varphi|^+$, therefore, we only study updating with the positive interpretation of a formula.

Conditioning belief as lower measure

If we look at belief as the lower approximation of the "real" probability function, then we know that the "real" probability function is in the set $\mathscr{M}_{\mathtt{Bel}}$. Therefore, to update the belief after learning that φ is the case, one can compute the Bayesian update of every probability in $\mathscr{M}_{\mathtt{Bel}}$. In that framework, this boils down to ignoring information states (that is, the elements $s \in S$) not supporting φ, on the other words, we just consider the sources that support φ. Here we have constraints in choosing the formulas that we can do the updating on them. In fact, based on Theorem 4.2.1, the belief function $\mathtt{Bel}$ defined on $P(S)$, is equal to $(\mathscr{M}_{\mathtt{Bel}})_*$, where $\mathscr{M}_{\mathtt{Bel}} = \{\mu : \mu(X) \geq \mathtt{Bel}(X),\ \text{for all } X \in P(S)\}$. Updating on a formula φ does make sense if $1 - \mathtt{Bel}(\overline{|\varphi|^+}) > 0$. Hence if $1 - \mathtt{Bel}(\overline{|\varphi|^+}) > 0$, then one can define the conditional belief on φ as follows: for every $X \in P(S)$,

$$\mathtt{Bel}_{|\varphi|^+}(X) = ((\mathscr{M}_{\mathtt{Bel}})_{|\varphi|^+})_*(X)$$

which gives us the following conditional belief function on formulas, for each $\psi \in \mathscr{L}_{\mathsf{BD}}$,

$$\mathtt{bel}^+_{|\varphi|^+}(\psi) = \mathtt{Bel}_{|\varphi|^+}(|\psi|^+) = ((\mathscr{M}_{\mathtt{Bel}})_{|\varphi|^+})_*(|\psi|^+).$$

In what follows, for sake of readability, we will write $\mathtt{bel}^+_{|\varphi|}$ and $\mathtt{Bel}_{|\varphi|}$ instead of $\mathtt{Bel}_{|\varphi|^+}$ and $\mathtt{bel}^+_{|\varphi|^+}$. Based on Theorem 4.2.2, we have the following explicit formula

$$\mathtt{bel}^+_{|\varphi|}(\psi) = \begin{cases} 1 & \text{if } \mathtt{Bel}(\overline{|\varphi|^+} \cup |\psi|^+) = 1, \\ \dfrac{\mathtt{Bel}(|\psi|^+ \cap |\varphi|^+)}{1 + \mathtt{Bel}(|\psi|^+ \cap |\varphi|^+) - \mathtt{Bel}(\overline{|\varphi|^+} \cup |\psi|^+)} & \text{if } \mathtt{Bel}(\overline{|\varphi|^+} \cup |\psi|^+) < 1. \end{cases}$$

Notice that since the update is performed on $\mathtt{Bel}$, both $\mathtt{bel}^+$ and $\mathtt{bel}^-$ are affected by the update. In addition, $\mathtt{bel}^+_{|\varphi|}(\varphi) = 1$ as expected. However, in general, $\mathtt{bel}^+_{|\varphi|}(\neg\varphi) \neq 0$, because $|\varphi|^+ \cap |\neg\varphi|^+ \neq \emptyset$.

Conditioning belief via mass functions

If we interpret belief as representing the information coming from pieces of evidence, then one can also update the belief function $\mathtt{Bel}$ via its associated mass function $\mathtt{m}_{\mathtt{Bel}}$ and Dempster-Shafer combination rule. We call that method DS conditioning. A piece of evidence fully supporting exactly φ is usually represented by the mass function $\mathtt{m}_{|\varphi|^+} : P(S) \to [0,1]$ such that $\mathtt{m}_{|\varphi|^+}(|\varphi|^+) = 1$ and $\mathtt{m}_{|\varphi|^+}(X) = 0$ otherwise. Therefore, the updating of $\mathtt{Bel}$ by finding positive information about φ, denoted $(\mathtt{Bel})^{|\varphi|}$, is the belief function associated to the mass function $\mathtt{m}_{\mathtt{Bel}} \oplus \mathtt{m}_{|\varphi|^+}$. Then, based on Proposition 4.2.2, we have:

Proposition 4.3.1. *The belief function* $(\mathtt{bel}^+)^{|\varphi|}$ *is defined if* $1 - \mathtt{Bel}(\overline{|\varphi|^+}) > 0$*, and, for every* $\psi \in \mathcal{L}_{\mathsf{BD}}$*,*

$$(\mathtt{bel}^+)^{|\varphi|}(\psi) = (\mathtt{Bel})^{|\varphi|}(|\psi|^+) = \frac{\mathtt{Bel}(|\psi|^+ \cup \overline{|\varphi|^+}) - \mathtt{Bel}(\overline{|\varphi|^+})}{1 - \mathtt{Bel}(\overline{|\varphi|^+})}.$$

It is well-known that DS combination rule is associative and commutative [Sha76], therefore DS conditioning of belief functions is commutative and associative as well. Notice that, for every $\varphi, \psi \in \mathcal{L}_{\mathsf{BD}}$, we have $\mathtt{m}_{|\varphi|^+} \oplus \mathtt{m}_{|\psi|^+} = \mathtt{m}_{|\psi|^+} \oplus \mathtt{m}_{|\varphi|^+} = \mathtt{m}_{|\varphi|^+ \cap |\psi|^+} = \mathtt{m}_{|\varphi \wedge \psi|^+}$, which implies that $((\mathtt{bel}^+)^{|\varphi|})^{|\psi|} = ((\mathtt{bel}^+)^{|\psi|})^{|\varphi|} = ((\mathtt{bel}^+)^{|\varphi \wedge \psi|})$. This means that, with DS conditioning, finding both a piece of information supporting φ and a piece of information supporting $\neg\varphi$ is equivalent to finding a contradictory piece of information about φ. Here again, notice that $(\mathtt{bel}^+)^{|\varphi|}(\neg\varphi)$ can be different than 0 because some states $s \in S$ can support both φ and $\neg\varphi$. In addition, it is worth noticing, that $\mathtt{Bel}_{|\varphi|}(X) \leq \mathtt{Bel}^{|\varphi|}(X)$ for every $X \subseteq S$ (see [Hal17, Theorem 3.8.6]). Therefore, $\mathtt{bel}^+_\varphi(X) \leq (\mathtt{bel}^+)^{|\varphi|}(X)$ and $\mathtt{bel}^-_\varphi(X) \leq (\mathtt{bel}^-)^{|\varphi|}(X)$.

4.3.3 Updating plausibility

Recall that the interpretation of φ and $\neg\varphi$ are independent, therefore, when we consider a mass function $\mathtt{m}$ and its associated belief and plausibility functions $\mathtt{bel}_\mathtt{m}$ and $\mathtt{pl}_\mathtt{m}$, it is often the case that $\mathtt{bel}_\mathtt{m}(\varphi) \not\leq \mathtt{pl}_\mathtt{m}(\varphi) = 1 - \mathtt{bel}_\mathtt{m}(\neg\varphi)$. Hence, if one wants to reason with a belief function and a plausibility function that provide an interval that contains the probability of φ, one needs to consider a belief and a plausibility function that are not associated to the same mass function, which we explained in the previous chapter in detail. Then, the question of updating the plausibility function directly, without going through its associated belief function arises. To do so, we introduce $\mathsf{DS}_{\mathtt{pl}}$ models: $\mathfrak{M} = \langle S, P(S), \mathtt{Bel}, \mathtt{Pl}, v^+, v^- \rangle$. As mentioned above, we can focus on updating based on positive information about a formula φ.

Conditioning plausibility as upper measure

Based on Proposition 4.2.1, $\mathtt{Pl}$ is an upper probability, that is, $\mathtt{Pl} = (\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})^*$. So again, one can define conditioning on a formula φ when $\mathtt{Pl}(\varphi) > 0$ as follows for every $\psi \in \mathscr{L}_{\mathsf{BD}}$:

$$\mathtt{pl}^+_{|\varphi|}(\psi) = ((\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})_{|\varphi|^+})^*(|\psi|^+). \tag{4.8}$$

From Lemma 4.2.2, we have the following explicit formula for updating plausibilities.

Proposition 4.3.2. *Let φ be a formula such that $\mathtt{Pl}(|\varphi|^+) > 0$,*

$$\mathtt{pl}^+_{|\varphi|}(\psi) = \frac{\mathtt{Pl}(|\psi|^+ \cap |\varphi|^+)}{\mathtt{Pl}(|\psi|^+ \cap |\varphi|^+) + \mathtt{Pl}(\overline{|\psi|^+} \cap |\varphi|^+)}$$

Notice that, as expected, $\mathtt{pl}^+_{|\varphi|}(\varphi) = 1$. But when we observe positive information about φ, $\mathtt{pl}^+$ is not the only belief function affected by this new piece of evidence, $\mathtt{pl}^-$ also changes. Here using the fact that $|\psi|^- = |\neg\psi|^+$, we have:

$$\mathtt{pl}^-_{|\varphi|}(\psi) = ((\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})_{|\varphi|^+})^*(|\psi|^-) = ((\mathscr{M}_{\mathtt{Bel}_{\mathtt{Pl}}})_{|\varphi|^+})^*(|\neg\psi|^+) = \mathtt{pl}^+_{|\varphi|}(\neg\psi).$$

Conditioning plausibility via mass function

DS conditioning can be applied to plausibility functions via their associated mass functions $\mathtt{m}_{\mathtt{pl}}$ (see Lemma 1.3.5). The mass function associated to the update of $\mathtt{Pl}$ based on some piece of information positively supporting φ is computed via Dempster-Shafer combination rule as follows: $\mathtt{m}_{\mathtt{pl}} \oplus \mathtt{m}_{|\varphi|^+}$. Based on Proposition 4.2.2, we get the following formula for the corresponding plausibility function $(\mathtt{pl}^+)^{|\varphi|}$ over formulas.

Proposition 4.3.3. *The function* $(\mathtt{pl}^+)^{|\varphi|}$ *is defined if* $\mathtt{Pl}(|\varphi|^+) > 0$*, and, for every* $\psi \in \mathscr{L}_{\mathsf{BD}}$*, we have*

$$(\mathtt{pl}^+)^{|\varphi|}(\psi) = \frac{\mathtt{Pl}(|\psi|^+ \cap |\varphi|^+)}{\mathtt{Pl}(|\varphi|^+)}.$$

4.4 Conclusion

This work presents methods to update belief and plausibility functions within the framework of BD logic. Recall that BD logic was introduced to reason about incomplete and contradictory information. In DS models, even though the underlying logic is non-classical, namely BD logic, the belief and plausibility functions are defined over the powerset of states. Therefore, we can import the techniques for updating belief functions from the classical logic literature. This work is in fact a first step. Indeed, we wish to look at belief and plausibility functions over De Morgan algebras to get a better understanding of the implication of combining belief functions and BD logic. Recall that De Morgan algebras provide the algebraic semantics for BD logic, and that there is no duality between BD models and De Morgan algebras. Therefore, there is no way to directly import our results on frames to De Morgan algebras. A natural first step will be to study the mathematical properties of belief and plausibility functions over De Morgan algebras, and to establish whether they can be represented as lower and upper probabilities over sets of non-standard probabilities. This would open various options to update the belief. Indeed, [KMR21] presents different ways to update non-standard probabilities, among which two ways that generalise Bayesian update. In addition, Dempster–Shafer combination rule can straight forwardly be transferred to De Morgan algebras (see [Bíl+22]) which provides a natural way to update belief functions over De Morgan algebras. However, it remains to be checked whether this method is equivalent to DS conditioning on DS models.

Conclusion and future work

In this thesis we have studied reasoning with incomplete and contradictory information. This topic has been studied at length regarding crisps pieces of information. However, handling contradictory evidence while dealing with graded pieces of information such as for instance probabilities remain a difficult question. A well established logic to reason with crisp incomplete and contradictory information is Belnap–Dunn logic. Here, we expand on the work in [KMR21] to integrate belief and plausibility functions to their proposal for probabilistic Belnap–Dunn models. This work is made within a research project aiming at developing a modular logical framework based on two-layered logics that would allow to formalise a wide variety of 'probabilistic reasoning'. The modularity is understood as having a framework able to handle agents whose reasoning is based on different logics (such as for instance classical logic, Belnap–Dunn, relevant logics...) and on different kinds of evidence (crisp, probabilistic, statistical, ...).

In [Bíl+20; Bíl+22], we lay the ground to this project and present a first formalisation of probabilistic reasoning over Belnap—Dunn logic and belief functions. However a lot remains to be done. [Bíl+22] leaves open questions on how to interpret plausibility. We could tackle this question by looking at relevant case studies, but also by relying on the correspondence between non-standard probabilities and four valued probabilities [Dun10; DK19; KMR21]. Another research direction to define more intuitive notions of belief and plausibility within the framework of Belnap—Dunn logic would be to work within a functionally complete expansion of Belnap—Dunn logic [SO14; OS15]. Indeed, Belnap—Dunn logic does not allow to talk about the set of information states that satisfy exactly p (that is that positively satisfy p and do not negatively satisfy $\neg p$) or that satisfy neither p nor $\neg p$, therefore when we define plausibility of a formula we cannot say that we want to consider the masses of the sets of states that contain at least a state that satisfy exactly p or that satisfies neither p nor $\neg p$.

In Chapter 4, we discuss conditional update on belief functions via the frame semantics. In this approach we start from a belief function on the powerset of states and use the non-classical valuations of the Belnp—Dunn model to define belief and plausibility over the formulas of Belnp—Dunn logic. An alternative strategy would be to directly define belief and plausibility functions on De Morgan algebras. With this approach combining belief functions via their associated mass functions and Dempster—Shafer combination rule is straight forward. But can lead to different results on the frames and on the algebras. Applying Dempster-Shafer's rule on the frames and on the algebras are not equivalent. On the other hand direct updating of probabilities, beliefs and plausibilities on the algebra based on lower envelopes, is still unknown for us. We have spent some time to understand these updating methods on the algebra and we think that investigating and comparing the two above methods on the algebra and on the set of states can be an interesting subject to continue. In fact, Studying the update directly on the algebra will probably help us to get a better insight on its interpretation within Belnap—Dunn logic.

In addition, as we mentioned earlier, we aim at getting a better understanding of reasoning with probabilistic (or graded) information over substructural logics. Therefore many questions remain to be studied. How to interpret more general notions of probabilities such as lower and upper probabilities [Dem67; WF82], inner and outer measures [Hal50] and credal sets [Lev83]? Which axioms for probabilities would make sense if we consider propositional logics with an implication?

My contributions

The main results and contributions of this manuscript are in Chapters 2,3 and 4 which are based on the following papers, respectively:

1. M. Bílková, S. Frittella, O. Majer, and S. Nazari. "Belief based on inconsistent information". In: International Workshop on Dynamic Logic. 2020, pp. 68–86.

2. M. Bílková, S. Frittella, D. Kozhemiachenko, O. Majer, and S. Nazari. "Reasoning with belief functions over Belnap–Dunn logic". In: preprint, arXiv:2203.01060.

3. S. Frittella, O. Majer, and S. Nazari. "Toward updating belief functions over Belnap–Dunn logic". In: accepted in Belief 2022, arXiv:2205.15159. (2022).

and also I have given and I will give the following presentations:

1. Probabilistic Reasoning Based on Incomplete and Inconsistent Information, Logica 2021, Hejnice, The Czech Republic, 27 September–1 October 2021.

2. Belief based on inconsistent information, Dynamic Logic: New Trends and Applications, Prague, the Czech Republic, October 2020.

3. Belief 2022, 7th International Conference on Belief Functions, October 26-28, 2022, Paris, France.

4. LIFA 2022, 31ème rencontre francophones sur la logique floue et ses application, du 20 au 21 octobre 2022, Toulouse France.

5. Non-standard probabilities and belief functions over Belnap-Dunn logic, Workshop On Non-Classical Logic And Probabilistic Reasoning, Bourges, France, 17-18 February

6. Belief based on inconsistent information, Dynamic Logic: New Trends and Applications, Prague, the Czech Republic, October 2020.

7. Belief based on inconsistent information: INSA Centre Val de Loire, LIFO, group seminar, Bourges, France, September 2021.

The first article was was the beginning of the project that we write in collaboration with Marta Bílková, Sabine Frittella, Ondrej Majer. we had many group discussion, we developed the case studies together, I developed the formalisms on the aggregation strategies, concerning section 3.2, I contributed to the discussions to understand which kind of logics we needed, but I did not contribute on the technical aspects of the logical formalisation. I had regular discussion with my coauthors, they gave me continuous feedback on my progress and advised me on technical and presentation issues, and we wrote the paper together.

The second paper which is a submitted almost a year ago to APAL (Annals of Pure and Applied Logic) written with the same co-authors plus Daniil Kozhemiachenko. I focused on studying the imprecise probability tools, belief and plausibility on the Belnap–Dunn logic. This work appears in the sections 2 and 3. The last section of the article presents logics to formalise reasoning with non-standard probabilities, belief and plausibility over Belnap–Dunn logic. I attended the discussion on that part of the project, but I did not contribute technically to this section. Concerning the first sections which I put in the manuscript, they are the result that we extract from weekly discussions. We edited them together and submitted the final version aggregated with the logical formalism.

The last article in which I focus on conditioning was my proposal and is accepted in the international peer reviewed conference Belief 2022. A French translation of the article is accepted in the francophone peer reviewed conference Rencontres francophones sur la logique floue et ses applications 2022. Sabine Frittella and Ondrej Majer helped me and gave me continuous feed-backs to complete the paper and edit the final version.

During all these papers, we had some other results that we did not have enough time to make them into papers, specially the two last papers contains many unanswered questions or partially answered questions that will be continued in appropriate time.

Bibliography

[ASV16] D. Aerts, S. Sozzo, and T. Veloz. “New fundamental evidence of non-classical structure in the combination of natural concepts”. In: *Philosophical Transactions of the Royal Society A* 374 (2016).

[Avr96] Arnon Avron. “The Structure of Interlaced Bilattices”. In: *Mathematical Structures in Computer Science* 6.3 (1996), pp. 287–299.

[Bar00] J.-P. Barthélemy. “Monotone functions on finite lattices: An ordinal approach to capacities, belief and necessity functions”. In: *Preferences and Decisions under Incomplete Knowledge*. Studies in fuzziness and soft computing. Heidelberg: Physica-Verlag HD, 2000, pp. 195–208.

[BC16] Juliana Bueno-Soler and Walter Carnielli. “Paraconsistent probabilities: consistency, contradictions and Bayes’ theorem”. In: *Entropy* 18.9 (2016), p. 325.

[BCH11] L. Běhounek, P. Cintula, and P. Hájek. “Introduction to Mathematical Fuzzy Logic”. In: *Handbook of Mathematical Fuzzy Logic*. Ed. by P. Cintula, P. Hájek, and C. Noguera. Vol. 37. Studies in logic. College Publications, 2011, pp. 1–102.

[BCN20a] P. Baldi, P. Cintula, and C. Noguera. “Classical and Fuzzy Two-Layered Modal Logics for Uncertainty: Translations and Proof-Theory”. In: *International Journal of Computational Intelligence Systems* 13.1 (2020), pp. 988–1001.

[BCN20b] P. Baldi, P. Cintula, and C. Noguera. “On Two-layered Modal Logics for Uncertainty”. Manuscript. 2020.

[Bel19] N.D. Belnap. “How a Computer Should Think”. In: *New Essays on Belnap-Dunn Logic*. Ed. by H. Omori and H. Wansing. Vol. 418. Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science). Springer, Cham, 2019.

[BFK21] Marta Bílková, Sabine Frittella, and Daniil Kozhemiachenko. “Constraint tableaux for two-dimensional fuzzy logics”. In: *Lecture Notes in Computer Science*. Lecture notes in computer science. Cham: Springer International Publishing, 2021, pp. 20–37.

[Bíl+20] M. Bílková, S. Frittella, O. Majer, and S. Nazari. “Belief based on inconsistent information”. In: *International Workshop on Dynamic Logic*. 2020, pp. 68–86.

[Bíl+22] M. Bílková, S. Frittella, D. Kozhemiachenko, O. Majer, and S. Nazari. “Reasoning with belief functions over Belnap–Dunn logic”. In: preprint, arXiv:2203.01060 (2022).

[CD19] Davide Ciucci and Didier Dubois. “A capacity-based framework encompassing Belnap–Dunn logic for reasoning about multisource information”. In: *International Journal of Approximate Reasoning* 106 (2019), pp. 107–127.

[CDK06] Chris Cornelis, Glad Deschrijver, and Etienne E. Kerre. “Advances and challenges in interval-valued fuzzy logic”. In: *Fuzzy Sets and Systems* 157 (2006), pp. 622–627.

[CGL18] M. Console, P. Guagliardo, and L. Libkin. “Propositional and Predicate Logics of Incomplete Information”. In: *Proceedings of KR 2018*. AAAI Press, Tempe, Arizona, 2018, pp. 592–601.

[CHN11] P. Cintula, P. Hájek, and C. Noguera. *Handbook of Mathematical Fuzzy Logic 1*. Studies in Logic 37. College Publications, 2011.

[CN14] Petr Cintula and Carles Noguera. “Modal Logics of Uncertainty with Two-Layer Syntax: A General Completeness Theorem”. In: *Proceedings of WoLLIC 2014*. 2014, pp. 124–136.

[Cod75] E. F. Codd. "Understanding relations (installment #7)". In: *FDT - Bulletin of ACM SIGMOD* 7.3 (1975), pp. 23–28.

[Dem67] A.P. Dempster. "Upper and lower probabilities induced by a multivalued mapping". In: *ann. math. stat.* 38.2 (Apr. 1967), pp. 325–339.

[Dem68] A.P. Dempster. "A generalization of Bayesian inference". In: *Journal of the Royal Statistical Society: Series B (Methodological)* 30.2 (1968), pp. 205–232.

[DK19] J Michael Dunn and Nicholas M Kiefer. "Contradictory information: Better than nothing? the paradox of the two firefighters". In: *Graham Priest on Dialetheism and Paraconsistency*. Springer, 2019, pp. 231–247.

[DL11] Antonio Di Nola and Ioana Leustean. "Łukasiewicz Logic and MV-Algebras". In: *Handbook of Mathematical Fuzzy Logic Vol. 2*. Ed. by P. Cintula, P. Hajek, and C. Noguera. College Publications, 2011.

[DP07] Didier Dubois and Henri Prade. "Possibility theory". In: *Scholarpedia* 2.10 (2007), p. 2074.

[DP88] D. Dubois and H. Prade. "Representation and combination of uncertainty with belief functions and possibility measures". en. In: *Computational Intelligence* 4.3 (Sept. 1988), pp. 244–264.

[Dun10] J Michael Dunn. "Contradictory information: Too much of a good thing". In: *Journal of Philosophical Logic* 39.4 (2010), pp. 425–452.

[Dun76] J. Michael Dunn. "Intuitive semantics for first-degree entailments and 'coupled trees'". In: *Philosophical Studies* 29.3 (1976), pp. 149–168.

[FHM90] R. Fagin, J. Y. Halpern, and N. Megiddo. "A logic for reasoning about probabilities". In: *Information and Computation* 87 (1990), pp. 78–128.

[Fit94] M. Fitting. "Kleene's three valued logics and their children". In: *Fundamenta informaticae* 20.1, 2, 3 (1994), pp. 113–131.

[FMN22] S. Frittella, O. Majer, and S. Nazari. "Toward updating belief functions over Belnap–Dunn logic". In: preprint, arXiv:2205.15159 (2022).

[Fon97] J.M. Font. “Belnap’s Four-Valued Logic and De Morgan Lattices”. In: *Logic Journal of the IGPL* 5.3 (1997), pp. 1–29. DOI: `10.1093/jigpal/5.3.1-e`.

[Fri+20a] S. Frittella, K. Manoorkar, A. Palmigiano, A. Tzimoulis, and N. Wijnberg. “Toward a Dempster-Shafer theory of concepts”. In: *International Journal of Approximate Reasoning* 125 (2020), pp. 14–25.

[Fri+20b] Sabine Frittella, Krishna Manoorkar, Alessandra Palmigiano, Apostolos Tzimoulis, and Nachoem Wijnberg. “Toward a Dempster-Shafer theory of concepts”. In: *International Journal of Approximate Reasoning* 125 (2020), pp. 14–25.

[Gin88] M.L. Ginsberg. “Multivalued logics: A uniform approach to reasoning in AI.” In: *Computer Intelligence* 4 (1988), pp. 256–316.

[GO15] A.; Grahne G.; Moallemi and A. Onet. “Intuitionistic data exchange”. In: *9th Alberto Mendelzon International Workshop on Foundations of Data Management*. 2015.

[Gra09] M. Grabisch. “Belief functions on lattices”. en. In: *International Journal of Intelligent Systems* 24.1 (Jan. 2009), pp. 76–95.

[Háj98] Petr Hájek. *Metamathematics of Fuzzy Logic*. Vol. 4. Trends in Logic. Kluwer, 1998.

[Hal17] J.Y. Halpern. *Reasoning about uncertainty*. 2nd. MIT press, 2017.

[Hal50] P. Halmos. *Measure Theory*. Van Nostrand Reinhold Co., New York, NY., 1950.

[Ham59] C. L. Hamblin. “The modal ‘probably’”. In: *Mind* 68 (1959), pp. 234–240.

[JR12] Ramon Jansana and Umberto Rivieccio. “Residuated bilattices”. In: *Soft Comput.* 16.3 (2012), pp. 493–504.

[Kle52] S. C. Kleene. *Introduction to metamathematics*. North-Holland, Amsterdam, 1952.

[KMR21] D. Klein, O. Majer, and S. Rafiee Rad. “Probabilities with gaps and gluts”. en. In: *Journal of Philosophical Logic* 50.5 (Oct. 2021), pp. 1107–1141.

[Knu73] Donald E. Knuth. “Fundamental Algorithms”. In: Addison-Wesley, 1973. Chap. 1.2.

[Lev83] Isaac Levi. *The enterprise of knowledge: An essay on knowledge, credal probability, and chance*. MIT press, 1983.

[Nel49] David Nelson. "Constructible Falsity". In: *The Journal of Symbolic Logic* 14.1 (1949), pp. 16–26. ISSN: 00224812.

[Odi08] S.P. Odintsov. *Constructive negations and paraconsistency*. Vol. 26. Trends in logic. Springer, 2008.

[OS15] H. Omori and K. Sano. "Generalizing Functional Completeness in Belnap-Dunn Logic". In: *Studia Logica* 103.5 (Oct. 2015), pp. 883–917. ISSN: 1572-8730. DOI: 10.1007/s11225-014-9597-5.

[OW17] H. Omori and H. Wansing. "40 years of FDE: An introductory overview". In: *Studia Logica* 105.6 (2017), pp. 1021–1049.

[PR13] A. Pietz and U. Rivieccio. "Nothing but the Truth". In: *Journal of Philosophical Logic* 42.1 (2013), pp. 125–135. ISSN: 1573-0433. DOI: 10.1007/s10992-011-9215-1. URL: https://doi.org/10.1007/s10992-011-9215-1.

[Pre18] A. Prenosil. "Reasoning with Inconsistent Information". PhD thesis. Charles University, Prague, 2018.

[Riv10] Umberto Rivieccio. "An Algebraic Study of Bilattice-based Logics". PhD thesis. University of Barcelona - University of Genoa, 2010.

[Sha76] Glenn Shafer. *A mathematical theory of evidence*. Princeton university press, 1976.

[SK94] P Smets and R Kennes. *The transferable belief model. Artificial Inteligence (66), No. 2*. 1994.

[SO14] K. Sano and H. Omori. "An expansion of first-order Belnap–Dunn logic". In: *Logic Journal of IGPL* 22.3 (2014), pp. 458–481.

[Sta11] R.P. Stanley. *Enumerative Combinatorics*. 2nd ed. Vol. 1. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.

[Vak77] D. Vakarelov. "Notes on N-lattices and constructive logic with strong negation". In: *Studia logica* 36.1–2 (1977), pp. 109–125.

[Wan08] H. Wansing. "Constructive negation, implication, and co-implication". In: *Journal of Applied Non-Classical Logics* 18.2–3 (2008), pp. 341–364. DOI: `10.3166/jancl.18.341-364`. URL: `https://doi.org/10.3166/jancl.18.341-364`.

[WF82] Peter Walley and Terrence L Fine. "Towards a frequentist theory of upper and lower probability". In: *The Annals of Statistics* 10.3 (1982), pp. 741–761.

[Zad82] Lotfi A Zadeh. "Test-score semantics for natural languages". In: *Coling 1982: Proceedings of the Ninth International Conference on Computational Linguistics*. 1982.

[Zho13] C. Zhou. "Belief functions on distributive lattices". In: *Artificial Intelligence* 201 (2013), pp. 1–31.

[Zho16] Chunlai Zhou. "Logical Foundations of Evidential Reasoning with Contradictory Information". In: *J. Michael Dunn on Information Based Logics*. Springer, 2016, pp. 213–246.